Moments of action provide insight into critical times for advection-diffusion-reaction processes

Adam J. Ellery,[1] Matthew J. Simpson,[1,2] Scott W. McCue,[1] and Ruth E. Baker[3]
[1]*School of Mathematical Sciences, Queensland University of Technology, Brisbane, Australia*
[2]*Tissue Repair and Regeneration Program, Institute of Health and Biomedical Innovation (IHBI), Queensland University of Technology, Brisbane, Australia*
[3]*Centre for Mathematical Biology, Mathematical Institute, University of Oxford, 24-29 St Giles', Oxford OX1 3LB, United Kingdom*
(Received 25 July 2012; revised manuscript received 4 September 2012; published 25 September 2012)

Berezhkovskii and co-workers introduced the concept of local accumulation time as a finite measure of the time required for the transient solution of a reaction-diffusion equation to effectively reach steady state [Biophys J. **99**, L59 (2010); Phys. Rev. E **83**, 051906 (2011)]. Berezhkovskii's approach is a particular application of the concept of mean action time (MAT) that was introduced previously by McNabb [IMA J. Appl. Math. **47**, 193 (1991)]. Here, we generalize these previous results by presenting a framework to calculate the MAT, as well as the higher moments, which we call the *moments of action*. The second moment is the variance of action time, the third moment is related to the skew of action time, and so on. We consider a general transition from some initial condition to an associated steady state for a one-dimensional linear advection-diffusion-reaction partial differential equation (PDE). Our results indicate that it is possible to solve for the moments of action exactly without requiring the transient solution of the PDE. We present specific examples that highlight potential weaknesses of previous studies that have considered the MAT alone without considering higher moments. Finally, we also provide a meaningful interpretation of the moments of action by presenting simulation results from a discrete random-walk model together with some analysis of the particle lifetime distribution. This work shows that the moments of action are identical to the moments of the particle lifetime distribution for certain transitions.

DOI: 10.1103/PhysRevE.86.031136 PACS number(s): 05.60.−k, 44.05.+e, 46.15.−x

I. INTRODUCTION

Estimating the time required for an advection-diffusion-reaction process to effectively reach steady state is important in many applications of physics, engineering, mathematics, and life sciences [1]. Such an estimate is called a critical time [2–5]. In 1991, McNabb and Wake [6,7] introduced the concept of mean action time (MAT) as a critical time for an industrial heat transfer problem and showed that the MAT can be determined without solving the partial differential equation (PDE) model for the transient solution.

More recently, Berezhkovskii and co-workers introduced a critical time, called the local accumulation time (LAT) [8–13], for a one-dimensional reaction-diffusion PDE as a model of morphogen gradient formation [8]. For a particular initial condition, $C_0(x)$, Berezhkovskii found the time-dependent solution, $C(x,t)$, the steady-state solution $C_\infty(x) = \lim_{t\to\infty} C(x,t)$, and the LAT [8–13]. This new tool allowed them to draw conclusions about whether morphogen gradients could provide positional information on biologically relevant time scales [8–13].

Our work showed that Berezhkovskii's definition of LAT is equivalent to McNabb's definition of MAT [1]. We extended the definition of MAT to apply to a general one-dimensional advection-diffusion-reaction PDE, and we derived a more general solution strategy to find an exact expression for the MAT. We analyzed a reaction-diffusion model in the context of a chemical reaction in a porous catalyst [14] and showed that the MAT of an advection-diffusion-reaction process is identical to the mean particle lifetime (MPLT) of the underlying random-walk process if the transition is uniform-to-uniform, meaning that both $C_0(x)$ and $C_\infty(x)$ are identically constant.

Here we generalize the work of McNabb [6,7], Berezhkovskii [8–13], and Ellery [1]. To draw an analogy with probability theory, we identify

$$F(t;x) = 1 - \frac{C(x,t) - C_\infty(x)}{C_0(x) - C_\infty(x)}, \quad t \geqslant 0, \tag{1}$$

as a cumulative distribution function with the properties $F = 0$ at $t = 0$ and $F \to 1^-$ as $t \to \infty$. Since we are interested in transitions from $C_0(x)$ to $C_\infty(x)$, we only consider the case in which $C_0(x) \neq C_\infty(x)$ and $F(t;x)$ is finite. Using Eq. (1), we obtain

$$f(t;x) = \frac{dF}{dt} = -\frac{\partial}{\partial t}\left[\frac{C(x,t) - C_\infty(x)}{C_0(x) - C_\infty(x)}\right] \tag{2}$$

as the probability density function. Thus we see that McNabb's definition of MAT,

$$T(x) = \int_0^\infty t f(t)\, dt, \tag{3}$$

is the expected value of $f(t)$. Here, we present a framework for calculating the MAT and higher moments by defining the *moments of action*,

$$M_n(x) = \int_0^\infty [t - T(x)]^n f(t)\, dt, \quad n \geqslant 2, \tag{4}$$

and noting that $M_2(x)$ is the variance of action time, $M_3(x)$ is related to the skew of action time, and so on. We show how to solve for $M_n(x)$ for a general one-dimensional advection-diffusion-reaction equation governing a transition from $C_0(x)$ to $C_\infty(x)$ without solving the underlying PDE model for the transient solution. Finally, we show that $M_n(x)$ can be interpreted as a moment of particle lifetime for certain transitions.

In summary, our results provide additional insight and practical information about the critical time for a general advection-diffusion-reaction process. This builds on previous studies which have neglected the higher moments and only considered the mean.

II. MOMENTS OF ACTION: LINEAR DIFFUSION

We begin by analyzing the linear diffusion equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2}, \quad 0 < x < L, \tag{5}$$

where $D > 0$ is the diffusivity. When we apply Eq. (5) to a situation with an initial condition, $C_0(x)$, and boundary conditions so that the solution evolves toward some steady state, $C_\infty(x) = \lim_{t\to\infty} C(x,t)$, the MAT is given by

$$T(x) = \frac{1}{f(x)}\int_0^\infty t\frac{\partial}{\partial t}[C(x,t) - C_\infty(x)]\,dt, \tag{6}$$

where $f(x) = C_\infty(x) - C_0(x)$. We have not explicitly given the boundary conditions for this transition and we will demonstrate how to calculate $T(x)$ and $M_n(x)$ with different boundary conditions for the PDE in Sec. IV. Integrating Eq. (6) by parts and noting that $C(x,t) - C_\infty(x)$ decays to zero exponentially fast as $t \to \infty$ [1], we obtain

$$T(x) = \frac{1}{f(x)}\int_0^\infty [C_\infty(x) - C(x,t)]\,dt. \tag{7}$$

Equation (7) is an expression for the MAT associated with an arbitrary transition from $C_0(x)$ to $C_\infty(x)$. Written in this form, the integral expression can be interpreted as the mean time required for the initial condition, $C_0(x)$, to asymptote to the steady state, $C_\infty(x)$.

To determine the second moment of action for Eq. (5), we apply Eq. (4) to obtain

$$M_2(x) = \frac{1}{f(x)}\int_0^\infty [t - T(x)]^2\frac{\partial}{\partial t}[C(x,t) - C_\infty(x)]\,dt. \tag{8}$$

To solve for $M_2(x)$, we expand the quadratic term in Eq. (8) and rewrite two of the three integrals in terms of $T(x)$. Applying integration by parts to the remaining integral on the right-hand side of Eq. (8), and noting that $C(x,t) - C_\infty(x)$ decays to zero exponentially fast as $t \to \infty$ [1], we obtain

$$M_2(x) + T^2(x) = \frac{2}{f(x)}\int_0^\infty t[C_\infty(x) - C(x,t)]\,dt. \tag{9}$$

The same procedure can be used to determine any higher moment such as $M_3(x)$, $M_4(x)$, and so on. The details of the solutions for each moment depend on $C_0(x)$ and $C_\infty(x)$.

For illustrative purposes, we begin with a spatially constant initial condition and steady state, $C_0(x) \equiv C_0$ and $C_\infty(x) \equiv C_\infty$. To find the MAT, we differentiate Eq. (7) twice with respect to x and combine the resulting expression with Eq. (5) to obtain $T''(x) = -1/D$, where we use primes to indicate differentiation with respect to x. With appropriate boundary conditions, $T(0) = T(L) = 0$, the solution is

$$T(x) = \frac{x(L-x)}{2D}. \tag{10}$$

With $C_0(x) \equiv C_0$ and $C_\infty(x) \equiv C_\infty$, we solve for $M_2(x)$ by differentiating Eq. (9) twice with respect to x. Combining it with the governing equation for $T(x)$ gives $M_2''(x) = -2T'(x)^2$ which, with $M_2(0) = M_2(L) = 0$, has the solution

$$M_2(x) = \frac{L^4 - (L-2x)^4}{96D^2}. \tag{11}$$

The fact that we have been able to arrive at an exact expression for $M_2(x)$ without solving Eq. (5) for $C(x,t)$ is a major attraction compared to other critical time definitions which require the full transient solution [2–5]. Given $T(x)$ and $M_2(x)$, we can quantify the spread about the mean. One way to do this is to use the coefficient of variation, $C_v(x) = M_2(x)^{1/2}/T(x)$ [15]. Taking the exponential distribution as a reference point with $C_v = 1$, we can classify distributions with $C_v < 1$ as low variance distributions, whereas distributions with $C_v > 1$ are classified as high variance distributions. Using such a definition, our results allow us to make a very simple, yet practical, distinction between high variance and low variance distributions in a way that has not been considered previously [1,6–11,13,16].

Our assumption of dealing with a spatially uniform initial condition and steady state has been made here for illustrative purposes only. In Sec. III, we present results for a general initial condition and steady state associated with a linear advection-diffusion-reaction equation.

III. MOMENTS OF ACTION: ADVECTION-DIFFUSION-REACTION PROCESSES

We now present a general approach for studying the moments of action for a one-dimensional linear advection-diffusion-reaction equation:

$$\frac{\partial C}{\partial t} = D\frac{\partial^2 C}{\partial x^2} - V\frac{\partial C}{\partial x} - kC, \quad 0 < x < L, \tag{12}$$

where V is the advective velocity and $k \geqslant 0$ is the reaction (or death) rate. We consider applying Eq. (12) to a general transition with initial condition, $C_0(x)$, and boundary conditions such that the solution evolves to a steady state, $C_\infty(x)$.

As we observed in Sec. II, applying Eq. (4) leads to a variable coefficient boundary value problem for $M_n(x)$, $n \geqslant 2$. Here we present a general transformation,

$$S_n(x) = \int_0^\infty t^n\frac{\partial C}{\partial t}\,dt = n\int_0^\infty t^{n-1}[C_\infty(x) - C(x,t)]\,dt, \quad n \geqslant 1, \tag{13}$$

that allows us to arrive at a constant coefficient boundary value problem. The integration in Eq. (13) makes use of the fact that $C(x,t) - C_\infty(x)$ decays to zero exponentially fast as $t \to \infty$ [1]. To obtain the differential equation for $S_n(x)$, we differentiate Eq. (13) twice with respect to x, multiply Eq. (12) by t^{n-1}, and integrate the resulting expression with respect to t from 0 to ∞. Combining these two expressions gives

$$S_n''(x) - \frac{V}{D}S_n'(x) - \frac{k}{D}S_n(x) = -\frac{n}{D}S_{n-1}, \quad n \geqslant 1, \quad \text{where}$$
$$S_0(x) = f(x), \quad S_1(x) = T(x)f(x), \quad \text{and}$$
$$S_n(x) = \int_0^\infty t^n\frac{\partial C}{\partial t}\,dt = M_n(x)f(x) - \sum_{k=0}^{n-1}\binom{n}{k}[-T(x)]^{n-k}S_k(x), \quad n \geqslant 2. \tag{14}$$

The transformation to the $S_n(x)$ variable allows the governing equation to be written as a constant coefficient ordinary differential equation which can always, in principle, be solved to give an exact solution. We will now present three particular transitions with different initial conditions and boundary conditions and show that it is straightforward to arrive at exact or numerical results for $T(x)$ and $M_2(x)$.

IV. RESULTS

A. Uniform-to-uniform transition: Diffusion only

Using the theory presented in Sec. II, we present some key results in Fig. 1 for the linear diffusion equation, Eq. (5). Results in Fig. 1(a) show a numerical solution of Eq. (5) with $C_0(x) \equiv 1$ to $C_\infty(x) \equiv 0$ that is obtained using an unconditionally stable implicit finite difference method with spatial discretization δx and a time step δt [17]. Figure 1(b) shows $T(x)$ and $M_2(x)^{1/2}$, both of which are spatially symmetric. It is also possible to arrive at exact expressions for $M_3(x)$ and $M_4(x)$. Instead, we now examine another transition that will highlight the importance of the moments of action.

B. Uniform-to-uniform transition: Advection-diffusion

We now calculate the moments of action for a transition from $C_0(x) \equiv 1$ to $C_\infty(x) \equiv 0$, associated with the advection-diffusion equation, Eq. (12), with $k = 0$. The first two moments are given by

$$T(x) = -\alpha e^{Vx/D} + \frac{x}{V} + \alpha,$$

$$M_2(x) = \alpha^2(3e^{VL/D} + e^{Vx/D})(1 - e^{Vx/D}) + \frac{4\alpha x}{V}e^{Vx/D} + \frac{2\alpha D}{V^2}(1 - e^{Vx/D}) + \frac{2Dx}{V^3}, \tag{15}$$

where $\alpha = L/(V[e^{VL/D} - 1])$. Profiles in Fig. 2(a) show the solution of Eq. (12) with $k = 0$, $D = 0.5$, and $V = -0.1$. The asymmetry of the density profiles reflects the presence of advective transport, and the corresponding $T(x)$ and $M_2(x)^{1/2}$ profiles in Fig. 2(b) are also asymmetric.

The importance of calculating the moments of action can be illustrated through the results in Fig. 2(b) since it is possible to identify different spatial locations with a different coefficient of variation. For example, at $x = 3$ and 30.33, we have $T(3) = 195.6$, $T(30.33) = 195.6$, $M_2(3)^{1/2} = 220.4$, and $M_2(30.33)^{1/2} = 137.0$, giving $C_v(3) = 1.13$ (high variance) and $C_v(30.33) = 0.70$ (low variance). Therefore, our results allow us to identify two distinct spatial locations with identical MAT, but very different spreads about the mean.

C. Uniform-to-nonuniform transition: Diffusion death

We now present details of a transition for a nonuniform steady state. We consider Eq. (12) with $V = 0$, on

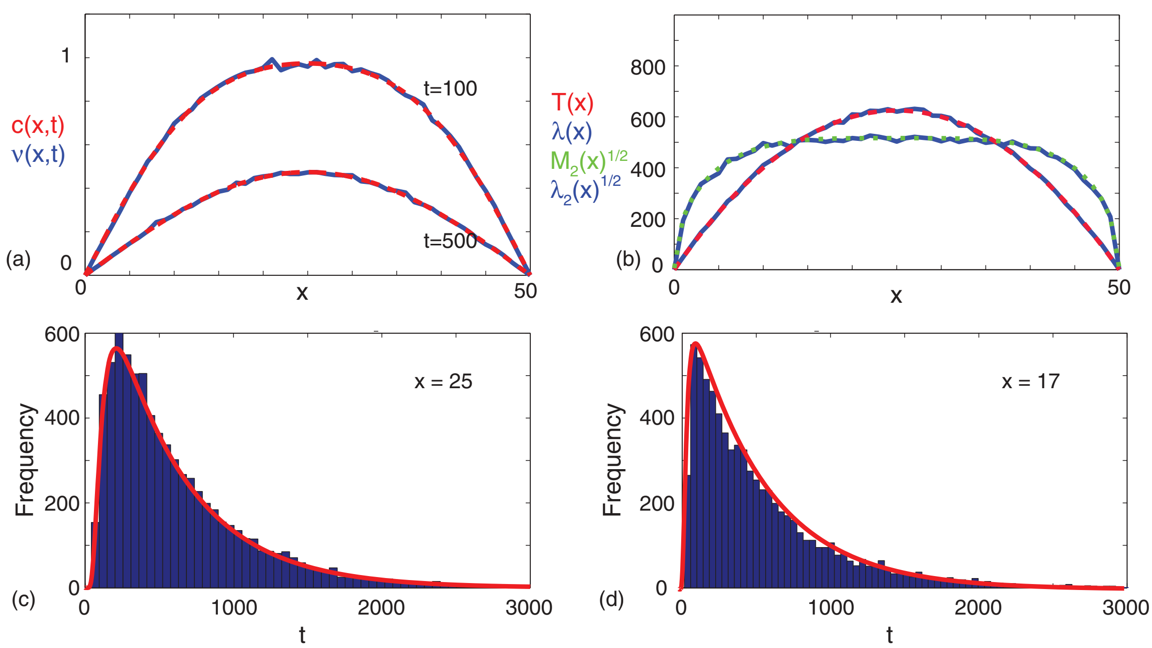

FIG. 1. (Color online) Moments of action for a uniform-to-uniform transition with linear diffusion on $0 < x < 50$. Profiles in (a) show the numerical solution of Eq. (12) with $D = 0.5$, $V = 0.0$, $k = 0.0$, $C_0(x) = 1$, $C(0,t) = 0$, and $C(50,t) = 0$, giving $C_\infty(x) = 0$. The numerical solution of Eq. (12), $C(x,t)$, is given in red (dashed) at $t = 100$ and 500, and was obtained using $\delta x = 0.01$ and variable time steps with $0.01 \leqslant \delta t \leqslant 1$. Discrete density profiles, $\nu(x,t)$, are superimposed in (a) at $t = 100$ and 500 in blue (solid). Discrete results were computed with $\Delta = 1$, $P_l = P_r = 0.5$, and $P_d = 0.0$. Results in (b) show the exact solution for $T(x)$ in red (dashed) and $M_2(x)^{1/2}$ in green (dotted), superimposed on $\lambda(x)$ and $\lambda_2(x)^{1/2}$ in blue (solid). Random-walk simulations in (a) and (b) were initiated with 7500 particles at each site. Histograms in (c) and (d) show the distribution of particle lifetime at $x = 25$ and 17, respectively. The red (solid) curve in (c) and (d) corresponds to $(-\partial C/\partial t)(\nu_{\max} t_b)$, where t_b is the histogram bin width and $\partial C/\partial t$ is obtained from the numerical solution of Eq. (12). In (c), $t_b = 60$, whereas in (d), $t_d = 45$.

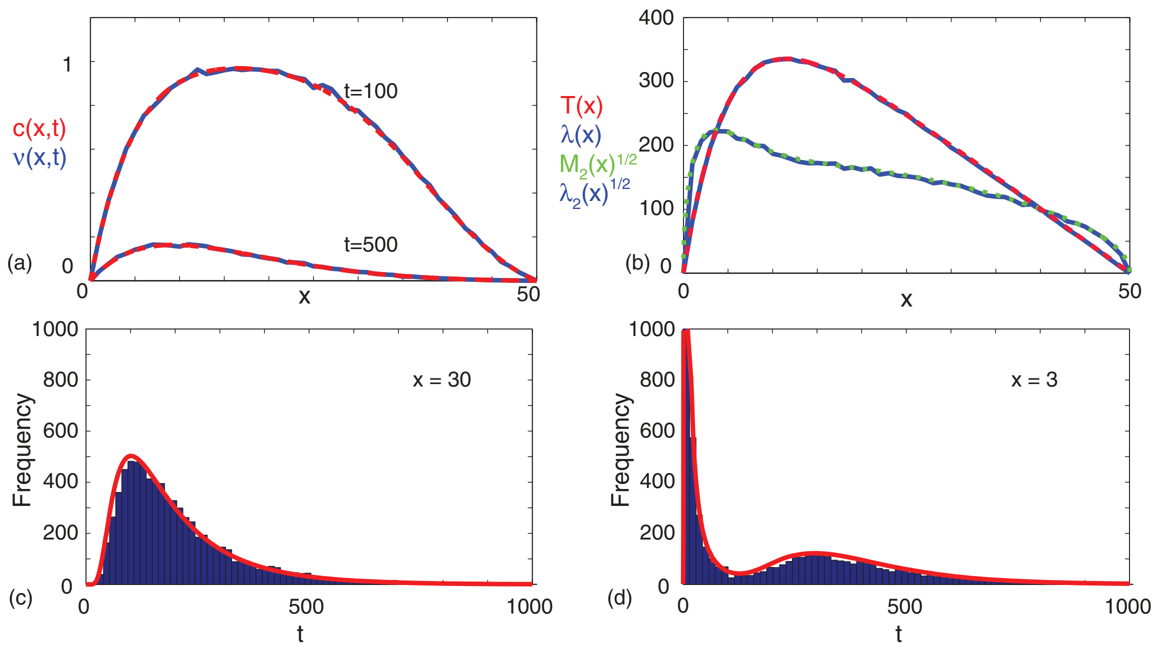

FIG. 2. (Color online) Moments of action for a uniform-to-uniform transition with linear advection-diffusion on $0 < x < 50$. Profiles in (a) show the numerical solution of Eq. (12) with $D = 0.5$, $V = -0.1$, $k = 0.0$, $C_0(x) = 1$, $C(0,t) = 0$, and $C(50,t) = 0$, giving $C_\infty(x) = 0$. The numerical solution of Eq. (12), $C(x,t)$, is given in red (dashed) at $t = 100$ and 500, and was obtained using $\delta x = 0.01$ and variable time steps with $0.01 \leqslant \delta t \leqslant 1$. Discrete density profiles, $\nu(x,t)$, are superimposed in (a) at $t = 100$ and 500 in blue (solid). Discrete results were computed with $\Delta = 1$, $P_l = 0.55$, $P_r = 0.45$, and $P_d = 0.0$. Results in (b) show the exact solution for $T(x)$ in red (dashed) and $M_2(x)^{1/2}$ in green (dotted), superimposed on $\lambda(x)$ and $\lambda_2(x)^{1/2}$ in blue (solid). Random-walk simulations in (a) and (b) were initiated with 7500 particles at each site. Histograms in (c) and (d) show the distribution of particle lifetime at $x = 3$ and 30, respectively, from the random-walk algorithm. The red (solid) curve in (c) and (d) corresponds to $(-\partial C/\partial t)(\nu_{\max} t_b)$, where t_b is the histogram bin width and $\partial C/\partial t$ is obtained from the numerical solution of Eq. (12). In (c), $t_b = 14$, whereas in (d), $t_d = 15$.

$0 < x < L$. The initial condition is $C_0(x) \equiv 0$, and the boundary conditions are $\partial C/\partial x = 0$ at $x = 0$ and $C(L,t) = 1$. This model is often used to represent a chemical reaction in a porous catalyst [14,18]. The steady state is $C_\infty(x) = \cosh(x\sqrt{k/D})/\cosh(L\sqrt{k/D})$. With appropriate boundary conditions, $T'(0) = 0$ and $T(L) = 0$, the MAT for this transition is

$$T(x) = \frac{x}{D\sqrt{k/D}(1 + e^{[2x\sqrt{k/D}]})} - \frac{x}{2D\sqrt{k/D}} + \frac{L(e^{[2L\sqrt{k/D}]} - 1)}{2D\sqrt{k/D}(e^{[2L\sqrt{k/D}]} + 1)}. \quad (16)$$

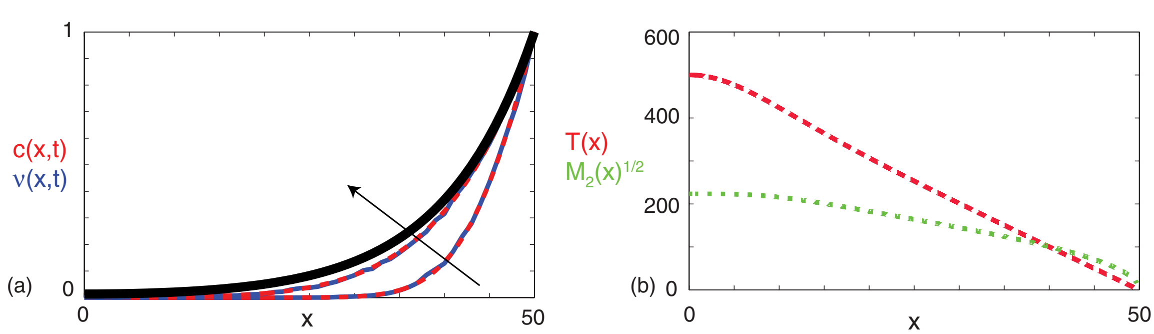

FIG. 3. (Color online) Moments of action for a uniform-to-nonuniform transition with linear reaction-diffusion on $0 < x < 50$. Results in (a) show the numerical solution of Eq. (12) with $D = 0.5$, $V = 0.0$, $k = 0.005$, $C(x,0) = 0$, $C(50,t) = 1$, and $\partial C/\partial x = 0$ at $x = 0$, giving $C_\infty(x) = \cosh(x\sqrt{k/D})/\cosh(L\sqrt{k/D})$. The numerical solution of Eq. (12), $C(x,t)$, is given in red (dashed) at $t = 50$ and 200, and was obtained using $\delta x = 0.01$ and variable time steps with $10^{-5} \leqslant \delta t \leqslant 1$. The arrow shows the direction of increasing time. The steady-state solution is given in black (thick solid). Discrete density profiles, $\nu(x,t)$, are superimposed in (a) at $t = 50$ and 200 in blue (thin solid). Discrete results were computed with $\Delta = 1$, $P_l = P_r = 0.5$, and $P_d = 0.005$. The Dirichlet boundary condition at $x = 50$ was represented by maintaining 7500 particles in the rightmost lattice site. Results in (b) show the exact solution for $T(x)$ in red (dashed) and a numerical solution for $M_2(x)^{1/2}$ in green (dotted) obtained using a central difference approximation with $\delta x = 0.005$.

To solve for $M_2(x)$ we need to solve Eq. (14) with $n = 2$, $V = 0$, and $T(x)$ given by Eq. (16). The algebraic expression for $M_2(x)$ is very lengthy and so we provide numerical results in Fig. 3 by solving Eq. (14) using central differences [17] with $S_2'(0) = 0$ and $S_2(L) = 0$. Profiles in Fig. 3(a) show a numerical solution of Eq. (12) with $k = 0.005$, $D = 0.5$, and $V = 0$. Figure 3(b) shows the exact solution for $T(x)$ and a numerical solution for $M_2(x)^{1/2}$.

V. INTERPRETATION IN TERMS OF PARTICLE LIFETIME

We now investigate the relationship between the moments of action and the moments of a particle lifetime by considering a biased random walk with death on a one-dimensional lattice with spacing Δ. Each site is at position $x = \Delta i$, where $i \in \mathbb{Z}^+$ is the site index. Time is uniformly discretized into intervals of duration τ. During each time interval, particles die with probability P_d and step to nearest-neighbor sites in the positive or negative x direction with probability P_r and P_l, respectively, with $P_l + P_r + P_d \leqslant 1$.

Using a Gillespie algorithm [19], we simulate the transitions in Figs. 1 and 2 with $\Delta = 1$ and each site initially occupied with $\nu_{\max}$ particles. As the algorithm proceeds, particles at the boundary sites are removed to represent the homogeneous Dirichlet boundary conditions. To simulate the transition in Fig. 3, we set $\Delta = 1$ and initialize each site with zero particles. As the algorithm proceeds, the number of particles at the rightmost site is maintained at $\nu_{\max}$ to represent the Dirichlet boundary condition. Particle density profiles are estimated using $\nu(x,t) = \hat{\nu}(x,t)/\nu_{\max}$, where $\hat{\nu}(x,t)$ is the number of particles at position x after time t.

We expect that the spatial distribution of particle density will, on average, be related to Eq. (12), where

$$D = \lim_{\Delta,\tau\to 0} \frac{\Delta^2(P_l + P_r)}{2\tau}, \quad V = \lim_{\Delta,\tau\to 0} \frac{\Delta(P_r - P_l)}{\tau},$$
$$k = \lim_{\Delta,\tau\to 0} \frac{P_d}{\tau}. \tag{17}$$

With $\Delta = 1$, we have $D = (P_l + P_r)/2$, $V = (P_r - P_l)$, and $k = P_d$, and our results in Figs. 1 and 3(a) confirm that the solution of Eq. (12) matches the discrete density profiles [20,21].

To estimate the MPLT for the results in Figs. 1 and 2, we record the time for each particle to exit the system. Averaging the particle lifetimes gives

$$\lambda(x) = \frac{1}{\nu_{\max}} \sum_{j=1}^{\nu_{\max}} t_j(x), \tag{18}$$

where $t_j(x)$ is the time taken for the jth particle that was originally placed at x to leave the system. The higher moments of particle lifetime can be estimated by calculating

$$\lambda_n(x) = \frac{1}{\nu_{\max}} \sum_{j=1}^{\nu_{\max}} [t_j(x) - \lambda(x)]^n, \quad n \geqslant 2. \tag{19}$$

Following [20,21], we provide an argument that leads to a system of discrete conservation equations describing the expected MPLT and higher moments of particle lifetime. We let $\mathbb{E}(T_{x_0})$ be the expected time taken before a particle is absorbed at a boundary or dies, given that the particle starts at location x_0. Here, we use the notation x_w to represent the location of a random walker after w time steps. To obtain an expression for $\mathbb{E}(T_{x_0})$, we condition on the first event being a motility event in either direction, a death, or a rest event:

$$\begin{aligned}\mathbb{E}(T_{x_0}) &= \sum_{t=0}^{\infty} t\, \mathbb{P}\big(T_{x_0} = t\big) \\ &= \sum_{t=0}^{\infty} t\big[\mathbb{P}\big(T_{x_0} = t \big| x_1 = x_0 + \Delta\big) P_r \\ &\quad + \mathbb{P}\big(T_{x_0} = t \big| x_1 = x_0 - \Delta\big) P_l + \tau P_d \\ &\quad + \mathbb{P}\big(T_{x_0} = t \big| x_1 = x_0\big)(1 - P_l - P_r - P_d)\big].\end{aligned} \tag{20}$$

Shifting the time index, $t = (t - \tau) + \tau$, we can express Eq. (20) in terms of $\mathbb{E}(T_{x_{0\pm\Delta}})$,

$$\begin{aligned}\mathbb{E}\big(T_{x_0}\big) &= \sum_{t=0}^{\infty}[(t-\tau)+\tau]\big[\mathbb{P}\big(T_{x_0+\Delta} = t-\tau\big)P_r + \mathbb{P}\big(T_{x_0-\Delta} = t-\tau\big)P_l + \tau P_d + \mathbb{P}\big(T_{x_0} = t-\tau\big)(1 - P_l - P_r - P_d)\big] \\ &= \big[\mathbb{E}\big(T_{x_0+\Delta}\big) + \tau\big]P_r + \big[\mathbb{E}\big(T_{x_0-\Delta}\big) + \tau\big]P_l + \big[\mathbb{E}\big(T_{x_0} + \tau\big)\big][1 - P_l - P_r - P_d] - \tau P_d.\end{aligned} \tag{21}$$

Equation (21) can be simplified to give

$$P_r \mathbb{E}\big(T_{x_0+\Delta}\big) - (P_l + P_r + P_d)\mathbb{E}\big(T_{x_0}\big) + P_l \mathbb{E}\big(T_{x_0-\Delta}\big) = -\tau. \tag{22}$$

For higher moments, starting with $n = 2$, denoted by $\mathbb{E}(T_{x_0}^2)$, we obtain

$$\begin{aligned}\mathbb{E}\big(T_{x_0}^2\big) &= \sum_{t=0}^{\infty} t^2\, \mathbb{P}\big(T_{x_0} = t\big) \\ &= \sum_{t=0}^{\infty}[(t-\tau)^2 + 2\tau(t-\tau) + \tau^2]\mathbb{P}\big(T_{x_0+\Delta} = t-\tau\big)P_r + \sum_{t=0}^{\infty}[(t-\tau)^2 + 2\tau(t-\tau) + \tau^2]\mathbb{P}\big(T_{x_0-\Delta} = t-\tau\big)P_l \\ &\quad + \sum_{t=0}^{\infty}[(t-\tau)^2 + 2\tau(t-\tau) + \tau^2]\mathbb{P}\big(T_{x_0} = t-\tau\big)(1 - P_l - P_r - P_d) + \tau^2 P_d.\end{aligned} \tag{23}$$

Rewriting the right-hand side of Eq. (23) and combining with Eq. (22) gives

$$P_r\mathbb{E}\left(T_{x_0+\Delta}^2\right) - (P_l + P_r + P_d)\mathbb{E}\left(T_{x_0}^2\right) + P_l\mathbb{E}\left(T_{x_0-\Delta}^2\right) = \tau^2 - 2\tau\mathbb{E}\left(T_{x_0}\right). \quad (24)$$

In general, for the nth moment, we obtain

$$P_r\mathbb{E}\left(T_{x_0+\Delta}^n\right) - (P_l + P_r + P_d)\mathbb{E}\left(T_{x_0}^n\right) + P_l\mathbb{E}\left(T_{x_0-\Delta}^n\right) = \sum_{k=0}^{n-1}\binom{n}{k}(-\tau)^{n-k}\mathbb{E}\left(T_{x_0}^k\right), \quad (25)$$

which holds for $n \geqslant 2$. This can be proved by induction.

To obtain a differential equation description, we identify $\mathbb{E}(T_{x_0}^n)$ with a continuous function $E_n(x)$ and expand in a Taylor series,

$$E_n(x \pm \Delta) = E_n(x) \pm \Delta E_n'(x) + \frac{\Delta^2}{2}E_n''(x) + O(\Delta^3). \quad (26)$$

Combining Eqs. (25) and (26), and taking limits as $\Delta \to 0$ and $\tau \to 0$ simultaneously, with the ratio (Δ^2/τ) held constant, we arrive at

$$E_n''(x) - \frac{V}{D}E_n'(x) - \frac{k}{D}E_n(x) = -\frac{n}{D}E_{n-1}(x), \quad (27)$$

where D, V, and k are given by Eq. (17).

In summary, Eq. (27) is equivalent to Eq. (14) under certain conditions. For transitions with $f'(x) \equiv 0$, we have $M_n(x) = S_n(x)$, which means that $M_n(x)$ is equivalent to $E_n(x)$. These quantities are not equivalent whenever $f'(x) \not\equiv 0$. Estimates of $\lambda_n(x)^{1/n}$, given in Figs. 1 and 2(b), provide a good approximation to $T(x)$ and $M_2(x)^{1/2}$, as expected. We also show histograms of $\lambda_1(x)$ in Figs. 1(c) and 1(d) and Figs. 2(c) and 2(d) to illustrate how the particle lifetime distribution varies at different locations in the domain. While the two histograms in Figs. 1(c) and 1(d) are similar in shape, we see that the histograms in Figs. 2(c) and 2(d) are very different in shape even though they have similar means.

Although several other studies have considered various aspects of the moments of particle lifetime [22–26], these previous studies did not highlight any link between these concepts and MAT.

VI. DISCUSSION AND CONCLUSION

Estimating the time required for the transient solution of an advection-diffusion-reaction equation to asymptote to the steady state is important for many applications [1,6–11,13,16]. Previous studies have used the concept of MAT to characterize the critical time without explicitly considering higher moments [1,6–11,13,16]. Here we provide a framework for calculating such higher moments which allows us to quantify the spread about the MAT. One practical way of doing this is to use the coefficient of variation as a way to distinguish between low variance ($C_v < 1$) and high variance ($C_v > 1$) distributions relative to the exponential distribution ($C_v = 1$).

Our analysis shows that the nth moment of action, $M_n(x)$, is governed by a variable coefficient boundary value problem that can be solved using a transformation outlined here. Details of two uniform-to-uniform transitions are used to show how to arrive at exact expressions for $T(x)$ and $M_2(x)$. Some expressions, especially the higher moments $M_3(x)$ and $M_4(x)$ (not shown), are complicated and best evaluated using symbolic computation. We also present details of a uniform-to-nonuniform transition for which we give the exact solution for $T(x)$, but we find that it is simpler, and faster, to evaluate $M_2(x)$ numerically.

We also present results from a stochastic random-walk algorithm and some analysis to show that the nth moment of action, $M_n(x)$, is equivalent to the nth moment of particle lifetime, $\lambda_n(x)$, provided that we consider a uniform-to-uniform transition with $f'(x) \equiv 0$ and $C_\infty(x) \equiv 0$. This result provides a useful physical interpretation of the meaning of $M_n(x)$ for a fairly general class of transitions. For transitions that are not uniform-to-uniform, where $f'(x) \not\equiv 0$, the correspondence between $M_n(x)$ and $\lambda_n(x)$ no longer holds.

Our work has demonstrated the key concepts of MAT and the moments of action associated with a one-dimensional advection-diffusion-reaction equation. Other practical examples would be two- and three-dimensional, or be modeled by a system of coupled differential equations, and we leave these points as open questions for future research.

ACKNOWLEDGMENTS

M.J.S. and R.E.B. acknowledge the Australian Research Council Discovery Project DP120100551 and the Royal Society 2011 International Exchange Scheme.

[1] A. J. Ellery, M. J. Simpson, S. W. McCue, and R. E. Baker, Phys. Rev. E **85**, 041135 (2012).
[2] R. I. Hickson, S. I. Barry, and G. N. Mercer, Int. J. Heat Mass Transf. **52**, 5776 (2009).
[3] R. I. Hickson, S. I. Barry, and G. N. Mercer, Int. J. Heat Mass Transf. **52**, 5784 (2009).
[4] R. I. Hickson, S. I. Barry, H. S. Sidhu, and G. N. Mercer, Int. J. Heat Mass Transf. **54**, 2624 (2011).
[5] R. I. Hickson, S. I. Barry, H. S. Sidhu, and G. N. Mercer, ANZIAM J. **52**, 333 (2011).
[6] A. McNabb and G. C. Wake, IMA J. Appl. Math. **47**, 193 (1991).
[7] A. McNabb, Math. Comput. Modell. **18**, 123 (1993).
[8] P. V. Gordon, C. Sample, A. M. Berezhkovskii, C. B. Muratov, and S. Y. Shvartsman, Proc. Natl. Acad. Sci. (USA) **108**, 6157 (2011).
[9] A. M. Berezhkovskii, C. Sample, and S. Y. Shvartsman, Biophys. J. **99**, L59 (2010).
[10] A. M. Berezhkovskii, C. Sample, S. Y. Shvartsman, Phys. Rev. E **83**, 051906 (2011).
[11] A. M. Berezhkovskii, J. Chem. Phys. **135**, 074112 (2011).
[12] A. M. Berezhkovskii and S. Y. Shvartsman, J. Chem. Phys. **135**, 154115 (2011).
[13] A. B. Kolomeisky, Phys. Chem. Lett. **2**, 1502 (2011).
[14] A. J. Ellery and M. J. Simpson, Chem. Eng. J. **169**, 313 (2011).

[15] L. H. Koopmans, D. B. Owen, and J. I. Rosenblatt, Biometrika **51**, 25 (1964).
[16] K. A. Landman and M. McGuinness, J. Appl. Math. Decision Sci. **4**, 125 (2000).
[17] M. J. Simpson, K. A. Landman, and T. P. Clement, Math. Comput. Sim. **70**, 44 (2005).
[18] E. Magyari, Chem. Eng. J. **143**, 161 (2008).
[19] D. T. Gillespie, J. Phys. Chem. **81**, 2340 (1977).
[20] S. Redner, *A Guide to First Passage Processes* (Cambridge University Press, Cambridge, 2001).
[21] G. Grimmett and D. Welsh, *Probability: An Introduction* (Oxford University Press, Oxford, 1986).
[22] A. M. Berezhkovskii, V. Zaloj, and N. Agmon, Phys. Rev. E **57**, 3937 (1998).
[23] A. Bar-Haim and J. Klafter, J. Chem. Phys. **109**, 5187 (1998).
[24] E. Barkai, J. Stat. Phys. **123**, 883 (2006).
[25] N. Agmon, Chem. Phys. Lett. **497**, 184 (2010).
[26] J.-S. Chen and W.-Y. Chang, J. Chem. Phys. **112**, 4723 (2000).