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Numerical solution of the time fractional reaction–diffusion equation with a moving boundary ☆

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ABSTRACT

A fractional reaction–diffusion model with a moving boundary is presented in this paper. An efficient numerical method is constructed to solve this moving boundary problem. Our method makes use of a finite difference approximation for the temporal discretization, and spectral approximation for the spatial discretization. The stability and convergence of the method is studied, and the errors of both the semi-discrete and fully-discrete schemes are derived. Numerical examples, motivated by problems from developmental biology, show a good agreement with the theoretical analysis and illustrate the efficiency of our method.

1. Introduction

In this paper, we shall investigate time fractional reaction–diffusion equation on a uniformly growing domain. The immobilized form is a class of fractional integro-differential equations given by

$$ {}_0^CD_t^{\gamma} u(t,x) = d(t)\partial_{xx}u(t,x) + Ku(t,x) + I_{0+}^{1-\gamma}\left[v(t,x)\partial_x u(t,x)\right] \tag{1.1} $$

for x in a fixed interval, where ${}_0^CD_t^{\gamma}$ denotes the Caputo fractional derivative, $I_{0+}^{1-\gamma}$ the Riemann–Liouville fractional integral and K is a reaction parameter. It is worthwhile to point out that the diffusive coefficient $d(t)$ depends on the moving boundary, and $v(t,x)$ denotes the velocity of domain growth.

Here, we first review some recent results on moving boundary problems. Moving boundary problems are mainly concerned with fluid flow in porous media and with diffusion and heat flow incorporating phase transformations or chemical reactions. Such problems are encountered in many industrial processes, for example, seepage through porous media, freezing

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or melting problems, and gas–solid reactions [1]. In reality, moving-boundary problems include both unknown boundary and prescribed-boundary problems. The former is often called a Stefan problem. For the unknown boundary moving boundary problem, one has to determine the motion of the interface together with the solution.

Exact solutions for moving boundary problems are only available under certain, limited circumstances. A similarity solution has been constructed for a prescribed-boundary problem in [2]. Muntean et al. [3] studied a two phase carbonation reaction model that has a moving unknown internal boundary, and presented the global existence and uniqueness of the solution.

Some efficient numerical methods have been presented for the classical Stefan problems, such as the spectral Petrov–Galerkin method [4,5], the finite element method [6], and the spectral element method [7]. For a prescribed-boundary moving boundary problem, Baines et al. proposed the moving mesh method [8], and Lee et al. studied the velocity-based moving mesh method [9]. In [10] the boundary element method was studied. The finite element method was employed to solve a model of vibrating elastic membrane in [11]. Gawlik et al. [12] reviewed some existing numerical method for prescribed-boundary moving boundary problems and proposed a high-order finite element method. In [13], a spectral method has also been studied. Additionally, the finite difference method was also applied to solve the prescribed-boundary moving boundary problems [14]. Recently, Yuan et al. [15] studied a three-dimensional moving boundary problem on the compressible miscible (oil and water) displacement by a second-order upwind difference fractional steps scheme applicable to parallel computing.

Over the past few decades, fractional differential equations have started to attract more and more attention. Fractional derivatives are extensively used as the tools for dealing with complex systems, such as anomalous diffusion, turbulence and amorphous material [16,17]. The non-local nature of fractional derivatives mean that these models are more suitable for studying history-related and time-related problems.

Based on fractional derivatives, the standard moving boundary problems have been extended in several areas of engineering and industry during the last few years. In [18], the authors presented a fractional anomalous diffusion model of drug release that is obtained by replacing the time derivative of the classical Stefan problem by the Caputo fractional derivative. In [19], a mathematical model containing the space-time fractional derivative was applied to model the melting and solidification process. Rajeev et al. presented a time fractional model of a generalized Stefan problem—a shoreline problem in [20]. Atkinson [21] also considered time fractional diffusion with a moving boundary and explicit results were obtained for the motion of planar, cylindrical, and spherical boundaries. In general, the numerical solution of fractional Stefan problem has been obtained by the homotopy perturbation method (for example see [19,20,22]). Recently, the finite difference method is also employed to solve the fractional Stefan problem [23].

In a biological system, domain expansion has been considered in classical Fickian reaction–diffusion models for biological pattern formation [24,25]. In [26] the authors studied the phenomenon of cellular migration on an underlying tissue, and examined the question of how long does it take for a wave of cells to colonize the whole tissue, or whether it is possible, while the tissue itself is expanding. More recently, Simpson et al. [27,28] extended the work in [26] by finding an exact solution of a linearized model of cell colonization in one, two and three dimensions.

In this paper, we shall investigate a linear fractional reaction–diffusion process describing anomalous diffusion on a growing domain with a moving boundary. The model, which extends the standard reaction–diffusion models [29], will be proposed in next section. In this model, the domain growth is determined by a local velocity $v(x,t)$. We focus mainly on the numerical method for such type of problem. Equation (1.1) can be derived by the use of a transformation converting the moving boundary into a fixed boundary problem. By the definition and properties of fractional derivative we can transform (1.1) into a fractional advection–reaction–diffusion equation

$$\partial_t w(t,x) + a(t,x)\partial_x w(t,x) = {}_0^{RL}D_t^{1-\gamma}[K_\gamma b(t)\partial_{xx} w(t,x)] + f(w,t,x). \tag{1.2}$$

In the simpler case where the model has constant coefficients, the solution of the fractional advection diffusion equation, a simple case of equation (1.2), has been investigated using a finite difference method [30]. Also, the variable coefficients space-time fractional advection diffusion equation has been studied by using the difference method in [31].

However, this is the first approach to solve equation (1.2) numerically. We shall use the finite difference method for time discretization and the spectral method for spatial discretization. Here, the main difficulty lies in two aspects. The first one is caused by the fact that the coefficients depend on the boundary function that leads to difficulties in the analysis of stability and convergence of the numerical approximation. The other arises from the convective term that is coupled with reactive term leads to more complexity in the temporal discretization.

The paper is arranged as follows. In next section, the fractional reaction–diffusion model with moving boundary is presented and the analytical solution is also presented for this problem with non-growing domain. The temporal discretization is considered in the third section. The stability and convergence of the semi-discrete scheme is derived in section 4. In section 5, the spectral approximation is analyzed and the full-discrete error is derived. In section 6, we consider the implementation of our method. Some examples are given in section 7 to show the efficiency and high order accuracy of our method. Finally, some remarks are given in section 8.

2. Mathematical model

We consider a conservation statement for a density, $C(t,x)$, and make the following assumptions:

- the direction and speed of migration is determined by a gradient;
- the tissue growth is independent of the cell density, $C(t,x)$;
- the system diffuses with a sub-diffusion and undergoes reaction at rate $R(C,t,x)$.

Application of the Reynolds transport theorem [24] for mass conservation for C on a domain $0 < x < L(t)$, gives

$$ {}_0^C D_t^\gamma C(t,x) + \partial_x(v(t,x)C(t,x)) = \mathcal{D}\partial_{xx}C(t,x) + R(C,t,x), $$

where the domain velocity $v(t,x)$ contributes an additional convective term. Here, $0 < \gamma < 1$, $\mathcal{D} > 0$ is the diffusivity coefficient, and ${}_0^C D_t^\gamma$ denotes the Caputo fractional derivative defined as

$$ {}_0^C D_t^\gamma C = \frac{1}{\Gamma(1-\gamma)} \int_0^t \frac{\partial_s C(s,x)}{(t-s)^\gamma} ds. $$

For a uniformly growing domain, the domain growth is associated with the velocity $v(t,x)$ by a relationship (see [26,29] for details)

$$ \frac{dL}{dt} = \int_0^{L(t)} \partial_x v dx = \int_0^{L(t)} \sigma(t) dx. \tag{2.1} $$

From (2.1), it follows that $\sigma(t) = \frac{L'(t)}{L(t)}$ and $v = x\sigma(t)$. In particular, for non-growing domain, $\sigma(t) \equiv 0$; for an exponentially growing domain with $L(t) = L(0)\exp(\alpha t)(\alpha > 0)$, $\sigma(t) = \alpha$; and for a linearly growing domain with $L(t) = L(0) + bt(b > 0)$, $\sigma(t) = \frac{b}{L(t)}$.

Now, we consider the linear fractional reaction–diffusion equation with moving boundary

$$ {}_0^C D_t^\gamma C(t,\xi) = \mathcal{D}\partial_{\xi\xi}C(t,\xi) - \partial_\xi(v(t,\xi)C(t,\xi)) + \mathcal{K}C(t,\xi),\ t > 0, 0 < \xi < L(t), \tag{2.2} $$

where $\mathcal{K}$ is a reaction parameter, $\mathcal{K} < 0$ implies a decay process and $\mathcal{K} > 0$ a growing process.

We assume the non-homogeneous Dirichlet boundary conditions:

$$ C(t,0) = C_l(t), C(t,L(t)) = C_r(t), $$

and the initial condition $C(0,\xi) = g(\xi)$

To solve the above problem, we introduce a coordinate transformation

$$ x = \xi / L(t). $$

Then, $v(t,\xi) = \xi\sigma(t) = xL'(t)$ for a uniformly growing domain. Let $u(t,x) = C(t,xL(t))$, and we have

$$ \partial_{\xi\xi}C = \frac{1}{L^2(t)}\partial_{xx}u,\ \partial_\xi(vC) = \frac{v}{L(t)}\partial_x u + \frac{u}{L(t)}\partial_x v, $$
$$ {}_0^C D_t^\gamma C(t,xL(t)) = {}_0^C D_t^\gamma u(t,x) - x \cdot \partial_x \left\{ I_{t+}^{1-\gamma}\left(u(t,x)\frac{L'(t)}{L(t)}\right)\right\}, $$

where I_{t+}^α denotes the Riemann–Liouville fractional integral defined as

$$ I_{t+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t \frac{f(s)}{(t-s)^{1-\alpha}} ds, \ \text{for } \alpha > 0. $$

Hence, the governing equation (2.2) transforms to

$$ {}_0^C D_t^\gamma u(t,x) = \frac{\mathcal{D}}{L^2(t)}\partial_{xx}u(t,x) - \sigma(t)x\partial_x u(t,x) + (\mathcal{K} - \sigma(t))u(t,x) + x \cdot \partial_x \left\{ I_{t+}^{1-\gamma}\left(u(t,x)\frac{L'(t)}{L(t)}\right)\right\} \tag{2.3} $$

and the zone including a moving boundary extends from $x = 0$ to $x = 1$. Therefore, we derive an immobilized boundary problem

$$ {}_0^C D_t^\gamma u(t,x) = d(t)\partial_{xx}u(t,x) - \sigma(t)x\partial_x u(t,x) + (\mathcal{K} - \sigma(t))u(t,x) + I_{t+}^{1-\gamma}[x\partial_x u(t,x)\sigma(t)], \quad t > 0,\ 0 < x < 1, \tag{2.4} $$

where $d(t) = \frac{\mathcal{D}}{L^2(t)}$, with the initial-boundary conditions $u(0,x) = u_0(x) = g(L(0)x)$ and boundary conditions

$$u(t,0) = u_l(t) = C(t,0),\ u(t,1) = u_r(t) = C(t,L(t)). \tag{2.5}$$

Further, for convenience, we shall convert problem (2.4)–(2.5) into one with homogeneous boundaries. To this goal, let $u(t,x) = v(t,x) + u_l(t) + (u_r(t) - u_l(t))x$, then (2.4) is clearly rewritten into one with homogeneous boundaries:

$${}_0^C D_t^\gamma u(t,x) = d(t)\partial_{xx}u(t,x) - \sigma(t)x\partial_x u(t,x) + (\mathcal{K} - \sigma(t))u(t,x) + I_{t+}^{1-\gamma}[x\partial_x u(t,x)\sigma(t)] + f(t,x), \tag{2.6}$$

where

$$\begin{aligned} f(t,x) = &- {}_0^C D_t^\gamma u_l(t) + (\mathcal{K} - \sigma(t))u_l(t) + (\mathcal{K} - 2\sigma(t))(u_r(t) - u_l(t))x \\ &- x \cdot {}_0^C D_t^\gamma [u_r(t) - u_l(t)] + I_{t+}^{1-\gamma}\{x\sigma(t)[u_r(t) - u_l(t)]\}. \end{aligned}$$

To end this section, we consider the analytical solution for the case of a non-growing domain. On a non-growing domain, the analytical solutions of time fractional partial differential equations [32] and multi-term time fractional partial differential equations [33] have been investigated by several authors. In this case, (2.4) is converted into

$${}_0^C D_t^\gamma u(t,x) = D\partial_{xx}u(t,x) + Ku(t,x) \tag{2.7}$$

with initial condition $u(0,x) = u_0(x)$ and boundary boundary condition $u(t,0) = u_l(t), u(t,1) = u_r(t)$.

Thus, by using the method of separation of variables (see [34] for example), the analytical solution of (2.7) can be derived. Let

$$\tilde{f}(t,x) = u_l(t) + [u_r(t) - u_l(t)]x - {}_0^C D_t^\gamma u_l(t) - x \cdot {}_0^C D_t^\gamma [u_r(t) - u_l(t)],$$

and expand $\tilde{f}(t,x)$ in the Fourier series

$$\tilde{f}(t,x) = \sum_{n=1}^{\infty} f_n(t) \sin(n\pi x).$$

Then, one can obtain the analytical solution of (2.7)

$$u(t,x) = \sum_{n=1}^{\infty} B_n(t) \sin(n\pi x) + u_l(t) + [u_r(t) - u_l(t)]x,$$

where

$$B_n(t) = 2E_{\gamma,1}\left[-(n^2\pi^2 D - K)t^\gamma\right]\int_0^1 v_0(x)\sin(n\pi x)dx + \int_0^t s^{\gamma-1}E_{\gamma,\gamma}\left[-(n^2\pi^2 D - K)s^\gamma\right] f_n(t-s)ds, \tag{2.8}$$

in which $v_0(x) = u_0(x) - u_l(0) - [u_r(0) - u_l(0)]x$ and $E_{\alpha,\beta}(t) = \sum_{n=0}^{\infty} \frac{t^n}{\Gamma(n\alpha+\beta)}$ is the Mittag–Leffler function based on the gamma function.

3. Semi-discrete approximation

Hereafter, we study the numerical solution of (2.6). For simplicity of analysis and without loss of generality, we consider the following equation

$${}_0^C D_t^\gamma u(t,x) = d(t)\partial_{xx}u(t,x) - \Upsilon u(t,x) + I_{t+}^{1-\gamma}[x\partial_x u(t,x)\sigma(t)] + f(t,x), \tag{3.1}$$

where $d(t) = \frac{1}{L^2(t)}$, $\sigma(t) = \frac{L'(t)}{L(t)}$ and $\Upsilon > 0$.

Let $t_k = k\tau, k = 0,1,2,\cdots,K$, where $\tau = \frac{T}{K}$ is the time step. By the definition of the Caputo derivative, equation (3.1) is recast as

$$I_{t+}^{1-\gamma}[\partial_t u - x\sigma(t)\partial_x u] = d(t)\partial_{xx}u - \Upsilon u + f(t,x). \tag{3.2}$$

Taking the Riemann–Liouville derivative of order $1-\gamma$ for both sides of (3.2) gives

$$\partial_t u - x\sigma(t)\partial_x u = {}_0^{RL}D_t^{1-\gamma}[d(t)\partial_{xx}u - \Upsilon u + f(t,x)], \tag{3.3}$$

where ${}^{RL}_{0}D_t^{1-\gamma}$ denotes the Riemann–Liouville fractional derivative defined by

$$ {}^{RL}_{0}D_t^{1-\gamma}h(t) = \frac{d}{dt}I_{0+}^{\gamma}h(t) = \frac{1}{\Gamma(\gamma)}\frac{d}{dt}\int_0^t \frac{h(s)}{(t-s)^{1-\gamma}}ds. $$

For the convenience of analysis, we recast (3.3) as

$$ \partial_t u - x\sigma(t)\partial_x u = {}^{RL}_{0}D_t^{1-\gamma}[d(t)\partial_{xx}u - \Upsilon u] + g(t,x), \tag{3.4} $$

here $g(t,x) = {}^{RL}_{0}D_t^{1-\gamma} f(t,x)$. It is worthwhile to stress that the limited regularity of g in time is enough to ensure the convergence rate of $O(\tau)$. Actually, the convergence of our method needs only $g \in L^1(0,T;L^2(\Lambda))$.

We first discretize the Riemann–Liouville fractional derivative

$$ {}^{RL}_{0}D_t^{1-\gamma}h(t_{k+1}) = \frac{I_{0+}^{\gamma}h(t_{k+1}) - I_{0+}^{\gamma}h(t_k)}{\tau} + O(\tau). \tag{3.5} $$

By the definition of Riemann–Liouville fractional integration, we have

$$ \begin{aligned} & I_{0+}^{\gamma}h(t_{k+1}) - I_{0+}^{\gamma}h(t_k) \\ &= \frac{1}{\Gamma(\gamma)}\int_0^{t_{k+1}}(t_{k+1}-s)^{\gamma-1}h(s)ds - \frac{1}{\Gamma(\gamma)}\int_0^{t_k}(t_k-s)^{\gamma-1}h(s)ds \\ &= \frac{1}{\Gamma(\gamma)}\int_0^{\tau}\frac{h(s)}{(t_{k+1}-s)^{1-\gamma}}ds - \frac{1}{\Gamma(\gamma)}\int_0^{t_k}\frac{h(s+\tau)-h(s)}{(t_k-s)^{1-\gamma}}ds \\ &= rb_k h(\tau) + R_{11}^{k+1}(\tau) + r\sum_{j=0}^{k-1} b_{k-j-1}[h(t_{j+2}) - h(t_{j+1})] + R_{12}^{k+1}(\tau) \\ &= rb_0 h(t_{k+1}) + r\sum_{j=0}^{k-1}(b_{j+1}-b_j)h(t_{k-j}) + R_1^{k+1}(\tau), \end{aligned} \tag{3.6} $$

where

$$ r = \frac{\tau^{\gamma}}{\Gamma(1+\gamma)},\ b_s = (s+1)^{\gamma} - s^{\gamma}, (s = 0,1,2,\cdots),\ R_1^{k+1}(\tau) = R_{11}^{k+1}(\tau) + R_{12}^{k+1}(\tau), \tag{3.7} $$

and

$$ R_{11}^{k+1}(\tau) = \frac{1}{\Gamma(\gamma)}\int_0^{\tau}\frac{h(s)-h(\tau)}{(t_{k+1}-s)^{1-\gamma}}ds, $$

$$ R_{12}^{k+1}(\tau) = \frac{1}{\Gamma(\gamma)}\sum_{j=0}^{k-1}\int_{t_j}^{t_{j+1}}\frac{[h(s+\tau)-h(s)] - [h(t_{j+2}) - h(t_{j+1})]}{(t_k-s)^{1-\gamma}}ds. $$

Lemma 3.1 *(See [35]). The coefficients b_k $(k = 0,1,2,\cdots)$ defined by (3.7) satisfy the following properties:*

i) $1 = b_0 > b_1 > b_2 > \cdots > 0$;
ii) *There exists a positive constant $\underline{\lambda}$, such that*

$$ \underline{\lambda}\tau \le b_k\tau^{\gamma},\ k = 1,2,\cdots. $$

Lemma 3.2. *Let $h(t) \in C^2([0,T])$, then*

i) $|R_{11}^{k+1}(\tau)| \le C_1 b_k \tau^{1+\gamma}$;
ii) $|R_{12}^{k+1}(\tau)| \le C_2\tau^2$,

where C_1, C_2 are the constants independent of τ and k.

Proof. The proof of this lemma can be found, in paper [35,36] for instance. □

Combining (3.6) with (3.5), we thus obtain the discretization equation of (3.4) in time as

$$\frac{u(t_{k+1},x)-u(t_k,x)}{\tau}=x\sigma(t_{k+1})\partial_x u(t_{k+1},x)+$$
$$\frac{r}{\tau}\left[d(t_{k+1})\partial_{xx}u(t_{k+1},x)+\sum_{j=0}^{k-1}(b_{j+1}-b_j)d(t_{k-j})\partial_{xx}u(t_{k-j},x)\right]$$
$$-\frac{r\Upsilon}{\tau}\left[u(t_{k+1},x)+\sum_{j=0}^{k-1}(b_{j+1}-b_j)u(t_{k-j},x)\right]+g(t_{k+1},x)+\frac{R_1^{k+1}(\tau,x)}{\tau}+O(\tau). \quad (3.8)$$

Set $R^{k+1}(\tau,x)=\frac{R_1^{k+1}(x)}{\tau}+O(\tau)$. It immediately follows that there exists a positive constant c_γ independent of τ and k, such that

$$|R^{k+1}(\tau,x)|\le c_\gamma b_k\tau^\gamma, \quad \forall x\in\Lambda \quad (3.9)$$

by Lemma 3.2 and Lemma 3.1. Denote by $u^{k+1}(x)$ the approximation solution of (3.8) for $u(t_{k+1},x)$, we derived the semi-discrete scheme of (3.4) as

$$u^{k+1}(x)=u^k(x)+\tau x\sigma_{k+1}\partial_x u^{k+1}(x)+r\Big[d_{k+1}\partial_{xx}u^{k+1}(x)$$
$$+\sum_{j=0}^{k-1}(b_{j+1}-b_j)d_{k-j}\partial_{xx}u^{k-j}(x)\Big]-r\Upsilon\left[u^{k+1}(x)+\sum_{j=0}^{k-1}(b_{j+1}-b_j)u^{k-j}(x)\right]+\tau g^{k+1}(x),\ k\ge 1 \quad (3.10)$$

and

$$u^1(x)=u^0(x)+\tau x\sigma_1\partial_x u^1(x)+rd_1\partial_{xx}u^1(x)-r\Upsilon u^1(x)+\tau g^1(x) \quad (3.11)$$

with

$$u^0(x)=u_0(x),\ d_j=d(t_j),\ \sigma_j=\sigma(t_j),\ g^j(x)=g(t_j,x),$$

for $j=1,2,\cdots,k+1$.

4. Stability and convergence

Set $\Lambda=(0,1)$, $\Omega=(0,T]\times(0,1)$. Denote by $(\cdot,\cdot)$ the inner product on the Hilbert space $L^2(\Lambda)$, by $\|\cdot\|_0$ the norm of $L^2(\Lambda)$.

Let $u^k(x), v^k(x) (k=1,2,\cdots)$ be the solutions of the semi-discrete approximation equations (3.10)–(3.11) associated to the initial values u_0, v_0, respectively. Let $e^k(x)=u^k(x)-v^k(x)$ for $k=0,1,2,\cdots$.

Theorem 4.1 *(Stability). Let $u_0, v_0\in L^2(\Lambda)$, $d(t)=1/L^2(t)$, $\sigma(t)=L'(t)/L(t)$. For the growing moving domain, where $\sigma(t)>0$, if τ satisfies*

$$\sum_{j=0}^{K-1}(b_j-b_{j+1})\left[\frac{L(T)}{L((K-j)\tau)}\right]^2\le 1, \quad (4.1)$$

then the semi-discrete scheme (3.10) is stable. Moreover, for any $k=1,2,\cdots,K$,

$$\|e^k\|_0\le\|e^0\|_0. \quad (4.2)$$

Proof. By (3.10) and (3.11), we have

$$e^1(x)=e^0(x)+\tau x\sigma_1\partial_x e^1(x)+rd_1\partial_{xx}e^1(x)-r\Upsilon e^1(x), \quad (4.3)$$
$$e^{k+1}(x)=e^k(x)+\tau x\sigma_{k+1}\partial_x e^{k+1}(x)+rd_{k+1}\partial_{xx}e^{k+1}(x)-r\Upsilon e^{k+1}(x)$$
$$+r\sum_{j=0}^{k-1}(b_{j+1}-b_j)\left[d_{k-j}\partial_{xx}e^{k-j}(x)-\Upsilon e^{k-j}(x)\right],\ \text{for } k\ge 1. \quad (4.4)$$

Multiplying the both sides of (4.2) by e^{k+1} and integrating over Λ, we get

$$(e^{k+1},e^{k+1})=(e^k,e^{k+1})+\tau\sigma_{k+1}(x\partial_x e^{k+1},e^{k+1})-rd_{k+1}(\partial_x e^{k+1},\partial_x e^{k+1})$$
$$-r\Upsilon(e^{k+1},e^{k+1})+r\sum_{j=0}^{k-1}(b_j-b_{j+1})d_{k-j}(\partial_x e^{k-j},\partial_x e^{k+1})+r\Upsilon\sum_{j=0}^{k-1}(b_j-b_{j+1})(e^{k-j},e^{k+1}). \quad (4.5)$$

Noting that

$$(x\partial_x e^{k+1},e^{k+1})=-\frac{1}{2}(e^{k+1},e^{k+1}),$$

and making use of the Cauchy–Schwartz inequality, we have

$$\|e^{k+1}\|_0^2\le\frac{1}{2}\|e^k\|_0^2+\frac{1}{2}\|e^{k+1}\|_0^2-\frac{1}{2}\tau\sigma_{k+1}\|e^{k+1}\|_0^2-rd_{k+1}\|\partial_x e^{k+1}\|_0^2$$
$$+\frac{r}{2}\sum_{j=0}^{k-1}(b_j-b_{j+1})d_{k-j}\left(\|\partial_x e^{k-j}\|_0^2+\|\partial_x e^{k+1}\|_0^2\right)$$
$$+\frac{r\Upsilon}{2}\sum_{j=0}^{k-1}(b_j-b_{j+1})\left(\|e^{k-j}\|_0^2+\|e^{k+1}\|_0^2\right)-r\Upsilon\|e^{k+1}\|_0^2. \quad (4.6)$$

By rearrangement, we have

$$\|e^{k+1}\|_0^2+r\sum_{j=0}^{k}b_jd_{k+1-j}\|\partial_x e^{k+1-j}\|_0^2+r\Upsilon\sum_{j=0}^{k}b_j\|e^{k+1-j}\|_0^2$$
$$\le\|e^k\|_0^2+r\sum_{j=0}^{k-1}b_jd_{k-j}\|\partial_x e^{k-j}\|_0^2+r\Upsilon\sum_{j=0}^{k-1}b_j\|e^{k-j}\|_0^2-\tau\sigma_{k+1}\|e^{k+1}\|_0^2$$
$$-rd_{k+1}\|\partial_x e^{k+1}\|_0^2+r\sum_{j=0}^{k-1}(b_j-b_{j+1})d_{k-j}\|\partial_x e^{k+1}\|_0^2+r\Upsilon\sum_{j=0}^{k-1}(b_j-b_{j+1})\|e^{k+1}\|_0^2-r\Upsilon\|e^{k+1}\|_0^2. \quad (4.7)$$

Let

$$E^n=\|e^n\|_0^2+r\sum_{j=0}^{n-1}b_jd_{n-j}\|\partial_x e^{n-j}\|_0^2+r\Upsilon\sum_{j=0}^{n-1}b_j\|e^{n-j}\|_0^2.$$

For the growing domain moving boundary problem, where $\sigma(t)>0$, then

$$-\tau\sigma_{k+1}\|e^{k+1}\|_0^2+r\Upsilon\sum_{j=0}^{k-1}(b_j-b_{j+1})\|e^{k+1}\|_0^2-r\Upsilon\|e^{k+1}\|_0^2\le 0.$$

On the other hand, noting that $d_j=1/L^2(j\tau)$, if

$$\sum_{j=0}^{k-1}(b_j-b_{j+1})\left[\frac{L((k+1)\tau)}{L((k-j)\tau)}\right]^2\le 1,$$

then

$$\sum_{j=0}^{k-1}(b_j-b_{j+1})d_{k-j}\le d_{k+1},$$

and it follows from (4.7) that

$$E^{k+1}\le E^k,\ \ \text{for } k\ge 1. \quad (4.8)$$

For $k=0$, we have from (4.3)

$$E^1\le E^0=\|e^0\|_0^2.$$

Making use of (4.8), we have for all $k \in \mathbb{N}$, $E^k \le \|e^0\|_0^2$. By the definition of E^k, the stability is derived. This completes the theorem. □

Remarks 4.1.

i) The semi-discrete approximation (3.10) is unconditionally stable for the non-growing boundary problem.
ii) For the uniform growing domain, the stability condition (4.1) can be solved to obtain the condition that the step size satisfies. For example, in the case of the exponentially growing domain with $L(t) = e^{\alpha t}$,

$$\sum_{j=0}^{K-1}(b_j - b_{j+1})\frac{e^{2\alpha T}}{e^{2\alpha(k-j)\tau}} = \sum_{j=0}^{K-1}(b_j - b_{j+1})e^{2\alpha(j+1)\tau} \le 1,$$

here, one can obtain τ by some numerical methods. Actually, due to

$$(b_j - b_{j+1})e^{2\alpha(j+1)\tau} \le e^{2\alpha T}(b_j - b_{j+1}),$$

it follows that $\sum_{j=0}^{\infty}(b_j - b_{j+1})e^{2\alpha(j+1)\tau}$ is convergent. Therefore, there exists τ_0 such that

$$\sum_{j=0}^{K-1}(b_j - b_{j+1})e^{2\alpha(j+1)\tau} \le \sum_{j=0}^{\infty}(b_j - b_{j+1})e^{2\alpha\tau_0(j+1)} \le 1$$

holds for any $\tau \le \tau_0$. The same result can be obtained for a linearly growing boundary.

Theorem 4.2 *(Convergence). Let $\sigma(t) \ge \underline{\theta} \ge 0$. Assume that $d(t), \partial_{xx}u(\cdot, x) \in C^2[0, T]$. Then, when the condition (4.1) holds the semi-discrete approximation (3.10) is convergent. Further, for any $k = 1, 2, \cdots$, there exists a positive constant C independent of τ and k, such that*

$$\|u(t_k, x) - u^k\|_0 \le C\tau.$$

Proof. Let $\eta^k(x) = u(t_k, x) - u^k(x)$. From (3.8) and (3.10), we have

$$\begin{aligned}\eta^{k+1}(x) &= \eta^k(x) + \tau x\sigma_{k+1}\partial_x\eta^{k+1}(x) + rd_{k+1}\partial_{xx}\eta^{k+1}(x) - r\Upsilon\eta^{k+1}(x)\\ &\quad + r\sum_{j=0}^{k-1}(b_{j+1} - b_j)\left[d_{k-j}\partial_{xx}\eta^{k-j}(x) - \Upsilon\eta^{k-j}(x)\right] + \tau R^{k+1}(x).\end{aligned} \tag{4.9}$$

Note that $\eta^k(0) = \eta^k(1) = 0$. Hence, the following equation still holds

$$\left(x\partial_x\eta^{k+1}, \eta^{k+1}\right) = -\frac{1}{2}\left(\eta^{k+1}, \eta^{k+1}\right).$$

Performing a similar proof to Theorem 4.1, we have

$$Y^{k+1} \le Y^k - rb_kd_{k+1}\|\partial_x\eta^{k+1}\|_0^2 - r\Upsilon b_k\|\eta^{k+1}\|_0^2 - \tau\sigma_{k+1}\|\eta^{k+1}\|_0^2 + 2\tau|(R^{k+1}, \eta^{k+1})|, \tag{4.10}$$

where

$$Y^k = \|\eta^k\|_0^2 + r\sum_{j=0}^{k-1}b_jd_{k-j}\|\partial_x\eta^{k-j}\|_0^2 + r\Upsilon\sum_{j=0}^{k-1}b_j\|\eta^{k-j}\|_0^2.$$

Therefore, by Young's inequality we obtain

$$\begin{aligned}Y^{k+1} &\le Y^k - r\Upsilon b_k\|\eta^{k+1}\|_0^2 + 2\tau|(R^{k+1}, \eta^{k+1})|\\ &\le Y^k - r\Upsilon b_k\|\eta^{k+1}\|_0^2 + \frac{\tau}{\epsilon}\|R^{k+1}\|_0^2 + \epsilon\tau\|\eta^{k+1}\|_0^2.\end{aligned} \tag{4.11}$$

Taking $\epsilon = \frac{\Upsilon b_k\tau^{\gamma-1}}{\Gamma(1+\gamma)}$, then it follows that

$$Y^{k+1} \le Y^k + \frac{\tau\Gamma(1+\gamma)}{\Upsilon b_k\tau^{\gamma-1}}\|R^{k+1}\|_0^2 \le Y^k + \frac{c_\gamma^2\Gamma(1+\gamma)\tau^2}{\Upsilon}b_k\tau^\gamma, \tag{4.12}$$

by using (3.9).

Noting that $Y^0 = \|\eta^0\|_0^2 = 0$, hence we obtain

$$Y^k \le \frac{c_\gamma^2 \Gamma(1+\gamma)\tau^2}{\Upsilon} \sum_{j=0}^{k-1} b_j \tau^\gamma \le \frac{c_\gamma^2 \Gamma(1+\gamma)\tau^2}{\Upsilon}(k\tau)^\gamma \le \frac{c_\gamma^2 \Gamma(1+\gamma)T^\gamma}{\Upsilon}\tau^2. \tag{4.13}$$

This finishes the proof of the theorem. □

5. Spectral approximation

In the following, we consider the full discretization scheme of (3.1) by the spectral method. For convenience, we introduce the following Sobolev norm

$$\|u\|_{1,\gamma}^2 = \|u\|_0^2 + \tau^\gamma \|\partial_x u\|_0^2 \text{ for } u \in H_0^1(\Lambda).$$

Additionally, we denote by $\|\cdot\|_m$ the standard Sobolev norm on space $H^m(\Lambda)$.

5.1. Variational formulation

Multiplying (3.10) by a test function $v \in H_0^1(\Lambda)$ and integrating, we obtain the variational formulation of problem (3.1): to find $u^{k+1}(k = 0, 1, \cdots) \in H_0^1(\Lambda)$, such that

$$\mathcal{B}(u^{k+1}, v) = \mathcal{F}^k(v),\ \forall v \in H_0^1(\Lambda), \tag{5.1}$$

where

$$\mathcal{B}(u^{k+1}, v) = (u^{k+1}, v) - \tau\sigma_{k+1}(x\partial_x u^{k+1}, v) + r d_{k+1}(\partial_x u^{k+1}, \partial_x v) + r\Upsilon(u^{k+1}, v),$$

$$\mathcal{F}^k(v) = (u^k, v) + r\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}(\partial_x u^{k-j}, \partial_x v) + r\Upsilon\sum_{j=0}^{k-1}(b_j - b_{j+1})(u^{k-j}, v) + \tau(g^{k+1}, v),$$

where g is defined as (3.4). Clearly, the bilinear operator $\mathcal{B}$ has the following properties:

Lemma 5.1. *For any $u^k, v \in H_0^1(\Lambda)$, there exist positive constant κ_1, κ_2 independent of u^k, v and τ, such that*

$$\mathcal{B}(u^k, u^k) \ge \kappa_1 \|u^k\|_{1,\gamma}^2, \quad \left|\mathcal{B}(u^k, v)\right| \le \kappa_2 \|u^k\|_{1,\gamma}\|v\|_{1,\gamma}.$$

5.2. Legendre spectral method

Let $P_N(\Lambda)$ denote the set of polynomials of degree N. Denote by $\mathring{P}_N(\Lambda) = H_0^1(\Lambda) \cap P_N(\Lambda)$.

The spatial discretization of the semi-discrete approximation (3.10) is to find $u_N^{k+1} \in \mathring{P}_N(\Lambda)(k = 0, 1, \cdots)$, such that

$$\mathcal{B}(u_N^{k+1}, v_N) = \mathcal{F}_N^k(v_N),\ \forall v_N \in \mathring{P}_N(\Lambda), \tag{5.2}$$

where

$$\mathcal{F}_N^k(v_N) = (u_N^k, v_N) + r\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}(\partial_x u_N^{k-j}, \partial_x v_N) + r\Upsilon\sum_{j=0}^{k-1}(b_j - b_{j+1})(u_N^{k-j}, v_N) + \tau(g^{k+1}, v_N).$$

In light of the coercivity and continuity of $\mathcal{B}$, the existence and uniqueness of the solution of (5.2) is assured by Lax–Milgram theorem. In what follows, we shall study the error estimate of the spectral approximation solution.

Let $\Pi_N : H_0^1(\Lambda) \to \mathring{P}_N(\Lambda)$ be the orthogonal projection on $\mathring{P}_N(\Lambda)$ in $H_0^1(\Lambda)$ such that

$$\mathcal{B}(\Pi_N u, v) = \mathcal{B}(u, v) \text{ for all } v \in \mathring{P}_N(\Lambda) \tag{5.3}$$

for any $u \in H_0^1(\Lambda)$. We have the error estimate (see Section 5.4 in [37])

$$\|u - \Pi_N u\|_{1,\gamma} \le \|u - \Pi_N u\|_1 \le CN^{1-m}\|u\|_m \tag{5.4}$$

for all $u \in H_0^m(\Lambda)$, with $m \ge 1$.

Theorem 5.2 *(Error estimate). Let $0 \le d(t) \le \overline{d}$, $\sigma(t) \ge \underline{\theta} > 0$. Assume that $u_0 \in L^2(\Lambda)$ and $\{u^k\}_{k=1}^K$ be the solutions of problem (5.1), $\{u_N^k\}_{k=1}^K$ the solutions of the full discrete problem (5.2). Suppose that $u^k \in H^m(\Lambda) \cap H_0^1(\Lambda)$, $m > 1$ and τ satisfies (4.1). Then, there exists a positive constant C independent of k, τ and N, such that*

$$\|u^k - u_N^k\|_{1,\gamma} \le C\tau^{-1}N^{1-m} \max_{0\le s\le K} \|u^s\|_m,$$

and

$$\|u^k - u_N^k\|_0 \le C\tau^{-1}N^{-m} \max_{0\le s\le K} \|u^s\|_m,$$

for all $k = 1, 2, \cdots, K$.

Proof. Let

$$\eta^j = u^j - \Pi_N u^j,\ \bar{\eta}^j = \Pi_N u^j - u_N^j,\ \epsilon^j = u^j - u_N^j = \eta^j + \bar{\eta}^j.$$

Subtracting (5.2) from (5.1), we have

$$\mathcal{B}(u^{k+1} - u_N^{k+1}, v_N) = (\epsilon^k, v_N) + r\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}(\partial_x\epsilon^{k-j}, \partial_x v_N) + r\Upsilon\sum_{j=0}^{k-1}(b_j - b_{j+1})(\epsilon^{k-j}, v_N) \tag{5.5}$$

for any $v_N \in \mathring{P}_N(\Lambda)$. In light of (5.3), $\mathcal{B}(u^{k+1} - u_N^{k+1}, v_N) = \mathcal{B}(\Pi_N u^{k+1} - u_N^{k+1}, v_N)$. Thus, let $v_N = \Pi_N u^{k+1} - u_N^{k+1}$, we obtain

$$\begin{aligned}
&\|\bar{\eta}^{k+1}\|_0^2 + rd_{k+1}\|\partial_x\bar{\eta}^{k+1}\|_0^2 + r\Upsilon\|\bar{\eta}^{k+1}\|_0^2 + \tau\sigma_{k+1}\|\bar{\eta}^{k+1}\|_0^2 \\
&\le \frac{1}{2}\|\epsilon^k\|_0^2 + \frac{1}{2}\|\bar{\eta}^{k+1}\|_0^2 + \frac{r}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}\left(\|\partial_x\epsilon^{k-j}\|_0^2 + \|\partial_x\bar{\eta}^{k+1}\|_0^2\right) \\
&\quad + \frac{r\Upsilon}{2}\sum_{j=0}^{k-1}(b_j - b_{j+1})\left(\|\epsilon^{k-j}\|_0^2 + \|\bar{\eta}^{k+1}\|_0^2\right),
\end{aligned} \tag{5.6}$$

namely,

$$\begin{aligned}
&\|\bar{\eta}^{k+1}\|_0^2 + rd_{k+1}\|\partial_x\bar{\eta}^{k+1}\|_0^2 + r\Upsilon\|\bar{\eta}^{k+1}\|_0^2 \\
&\le \|\epsilon^k\|_0^2 + r\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}\|\partial_x\epsilon^{k-j}\|_0^2 + r\Upsilon\sum_{j=0}^{k-1}(b_j - b_{j+1})\|\epsilon^{k-j}\|_0^2 \\
&\quad - rb_k d_{k+1}\|\partial_x\bar{\eta}^{k+1}\|_0^2 - rb_k\Upsilon\|\bar{\eta}^{k+1}\|_0^2 - \tau\sigma_{k+1}\|\bar{\eta}^{k+1}\|_0^2.
\end{aligned} \tag{5.7}$$

Noting that

$$\|\bar{\eta}^{k+1}\|_0^2 = \|\epsilon^{k+1}\|_0^2 - \|\eta^{k+1}\|_0^2 - 2(\eta^{k+1}, \bar{\eta}^{k+1}),$$

$$\|\partial_x\bar{\eta}^{k+1}\|_0^2 = \|\partial_x\epsilon^{k+1}\|_0^2 - \|\partial_x\eta^{k+1}\|_0^2 - 2(\partial_x\eta^{k+1}, \partial_x\bar{\eta}^{k+1}).$$

Therefore, we have

$$\begin{aligned}
\mathcal{E}^{k+1} &\le \mathcal{E}^k + \|\eta^{k+1}\|_0^2 + rd_{k+1}\|\partial_x\eta^{k+1}\|_0^2 + r\Upsilon\|\eta^{k+1}\|_0^2 \\
&\quad + 2(\eta^{k+1}, \bar{\eta}^{k+1}) + 2rd_{k+1}(\partial_x\eta^{k+1}, \partial_x\bar{\eta}^{k+1}) + 2r\Upsilon(\eta^{k+1}, \bar{\eta}^{k+1}) \\
&\quad - \tau\underline{\theta}\|\bar{\eta}^{k+1}\|_0^2 - rb_k d_{k+1}\|\partial_x\bar{\eta}^{k+1}\|_0^2 - rb_k\Upsilon\|\bar{\eta}^{k+1}\|_0^2,
\end{aligned} \tag{5.8}$$

where

$$\mathcal{E}^n = \|\epsilon^n\|_0^2 + r\sum_{j=0}^{n-1}b_j d_{n-j}\|\partial_x\epsilon^{n-j}\|_0^2 + r\Upsilon\sum_{j=0}^{n-1}b_j\|\epsilon^{n-j}\|_0^2.$$

By Young's inequality,

$$2(\eta^{k+1},\bar{\eta}^{k+1}) \le \tau\underline{\theta}\|\bar{\eta}^{k+1}\|_0^2 + \frac{1}{\tau\underline{\theta}}\|\eta^{k+1}\|_0^2,$$

$$2rd_{k+1}(\partial_x\eta^{k+1},\partial_x\bar{\eta}^{k+1}) \le rb_k d_{k+1}\|\partial_x\bar{\eta}^{k+1}\|_0^2 + \frac{rd_{k+1}}{b_k}\|\partial_x\eta^{k+1}\|_0^2,$$

$$2r\Upsilon(\eta^{k+1},\bar{\eta}^{k+1}) \le rb_k\Upsilon\|\bar{\eta}^{k+1}\|_0^2 + \frac{r\Upsilon}{b_k}\|\eta^{k+1}\|_0^2.$$

Hence, it yields that by (5.8)

$$\begin{aligned}\mathcal{E}^{k+1} &\le \mathcal{E}^k + \|\eta^{k+1}\|_0^2 + rd_{k+1}\|\partial_x\eta^{k+1}\|_0^2 + r\Upsilon\|\eta^{k+1}\|_0^2 + \frac{1}{\tau\underline{\theta}}\|\eta^{k+1}\|_0^2 + \frac{rd_{k+1}}{b_k}\|\partial_x\eta^{k+1}\|_0^2 + \frac{r\Upsilon}{b_k}\|\eta^{k+1}\|_0^2 \\ &\le \mathcal{E}^k + C\tau^{-1}\max_{1\le s\le K}\|\eta^s\|_{1,\gamma}^2 \end{aligned} \tag{5.9}$$

since $\frac{1}{b_k} \le \frac{\tau^{\gamma-1}}{\underline{\lambda}}$ by Lemma 3.1. Here, C is a positive constant that depends only on $\underline{\theta}, \underline{\lambda}, \gamma, \Upsilon$ and $\overline{d}$.

Making use of (5.9) and $\mathcal{E}^0 = 0$, we have

$$\mathcal{E}^k \le Ck\tau^{-1}\max_{1\le s\le K}\|\eta^s\|_{1,\gamma}^2 \le CT\tau^{-2}\max_{1\le s\le K}\|\eta^s\|_{1,\gamma}^2.$$

By the definition of $\mathcal{E}^k$, we obtain

$$\|u^k - u_N^k\|_{1,\gamma} \le C\tau^{-1}\max_{1\le s\le K}\|u^s - \Pi_N u^s\|_{1,\gamma}.$$

The first estimate of Theorem 5.2 is derived.

Now, we prove the second error estimate in $L^2(\Lambda)$ sense by using the duality argument. By the basic theory of elliptic equation and properties of $\mathcal{B}$, it follows that for any $\psi \in L^2(\Lambda)$, the following equation

$$-rd_{k+1}\partial_{xx}u - \tau\sigma_{k+1}x\partial_x u + (1+r\Upsilon)u = \psi \tag{5.10}$$

has a unique solution $u \in H^2(\Lambda)\cap H_0^1(\Lambda)$ and $\|u\|_2 \le C\|\psi\|_0$. Let v be the solution of the dual problem of (5.10), then $\|v\|_2 \le C\|\psi\|_0$, and the following holds

$$\mathcal{B}(z,v) = (\psi, z) \text{ for any } z \in H_0^1(\Lambda).$$

Taking $z = u^{k+1} - u_N^{k+1}$, then by Lemma 5.1

$$\begin{aligned}&(\psi, u^{k+1} - u_N^{k+1}) = \mathcal{B}(u^{k+1} - u_N^{k+1}, v - \Pi_N v)\\ &\le \kappa_2\|u^{k+1} - u_N^{k+1}\|_{1,\gamma}\|v - \Pi_N v\|_{1,\gamma}\\ &\le CN^{-1}\|u^{k+1} - u_N^{k+1}\|\|v\|_2\\ &\le CN^{-1}\|u^{k+1} - u_N^{k+1}\|_{1,\gamma}\|\psi\|_0.\end{aligned}$$

Therefore, we deduce that

$$\begin{aligned}&\|u^{k+1} - u_N^{k+1}\|_0 = \sup_{\psi\in L^2(\Lambda)}\frac{|(u^{k+1} - u_N^{k+1},\psi)|}{\|\psi\|_0}\\ &\le CN^{-1}\|u^{k+1} - u_N^{k+1}\|_{1,\gamma}.\end{aligned}$$

This finishes the proof of the theorem. □

Remark 5.1. This theorem still holds for $\underline{\theta} = 0$. It needs some modifications by Young's inequality to get an estimate.

Finally, we have the following error estimate by combining Theorem 4.2 with Theorem 5.2.

Theorem 5.3. *Let $d(t) \ge 0$ be increasing in t, $\sigma(t) \ge \underline{\theta} \ge 0$. Assume that $u_0 \in L^2(\Lambda)$ and $d(t) \in C^2[0,T]$. If $u \in C^2(0,T;H^2(\Lambda)\cap H_0^1(\Lambda))\cap L^\infty(0,T;H^m(\Lambda))$ $(m>1)$ is the solution of (3.4), and τ satisfies (4.1), then*

$$\|u(t_k,x) - u_N^k\|_0 \le C(\tau + \tau^{-1}N^{-m}),$$

where C is a constant independent of k, τ and N. □

5.3. An improved estimate

The coefficient τ^{-1} ahead of the spectral approximation estimate may be dropped by following the line in [38]. However, only an estimate of order N^{2-m} is obtained.

Theorem 5.4 *(Improved error bound). Let $d(t) \geq 0$, $\sigma(t) \geq \underline{\theta} > 0$. Assume that $u_0 \in L^2(\Lambda)$ and $\{u_N^k\}_{k=1}^K$ the solutions of the full discrete problem (5.2). Suppose that $u \in H^m(\Lambda) \cap H_0^1(\Lambda)$, $m > 1$ and τ satisfies (4.1). Then, there exists a positive constant C independent of k, τ and N, such that*

$$\|u(t_k, x) - u_N^k\|_0 \leq C(\tau + N^{2-m}),$$

for all $k = 1, 2, \cdots, K$.

Proof. Denote

$$\partial_t u(t_{k+1}, x) = \frac{1}{\tau}\left[\Pi_N u(t_{k+1}, x) - \Pi_N u(t_k, x)\right] + R_{21}^{k+1}(\tau, x),$$

where

$$R_{21}^{k+1}(\tau, x) = (I - \Pi_N)\partial_t u(t_{k+1}, x) + \Pi_N \left[\partial_t u(t_{k+1}, x) - \frac{u(t_{k+1}, x) - u(t_k, x)}{\tau}\right].$$

Let

$$R_{22}^{k+1}(\tau, x) = \partial_x u(t_{k+1}, x) - \partial_x \Pi_N u(t_{k+1}, x),$$
$$R_{23}^{k+1}(\tau, x) = \partial_{xx} u(t_{k+1}, x) - \partial_{xx} \Pi_N u(t_{k+1}, x).$$

Then,

$$\|R_{21}^{k+1}\|_0 = O(\tau + N^{-m}), \ \|R_{22}^{k+1}\|_0 = O(N^{1-m}), \ \|R_{23}^{k+1}\|_0 = O(N^{2-m}).$$

Similar to (3.8), we have

$$\begin{aligned} \frac{\Pi_N u(t_{k+1}, x) - \Pi_N u(t_k, x)}{\tau} &= x\sigma(t_{k+1})\partial_x \Pi_N u(t_{k+1}, x) + \\ \frac{r}{\tau}&\left[d(t_{k+1})\partial_{xx}\Pi_N u(t_{k+1}, x) + \sum_{j=0}^{k-1}(b_{j+1} - b_j)d(t_{k-j})\partial_{xx}\Pi_N u(t_{k-j}, x)\right] \\ -\frac{r\Upsilon}{\tau}&\left[\Pi_N u(t_{k+1}, x) + \sum_{j=0}^{k-1}(b_{j+1} - b_j)\Pi_N u(t_{k-j}, x)\right] + g(t_{k+1}, x) + R_2^{k+1}(\tau, x), \end{aligned} \tag{5.11}$$

where $R_2^{k+1}(\tau, x) = R_{21}^{k+1}(\tau, x) + R_{22}^{k+1}(\tau, x) + R_{23}^{k+1}(\tau, x) + R^{k+1}(\tau, x)$. In light of (3.9), one has

$$|R_2^{k+1}(\tau, x)| \leq c_\gamma b_k \tau^\gamma + C(\tau + N^{2-m}), \quad \forall x \in \Lambda \tag{5.12}$$

where C is a constant independent of τ, k and N. Therefore, for any $v_N \in \mathring{P}_N(\Lambda)$

$$\begin{aligned} (\Pi_N u(t_{k+1}, x), v_N) &= (\Pi_N u(t_k, x), v_N) + \tau\left(x\sigma_{k+1}\partial_x \Pi_N u(t_{k+1}, x), v_N\right) \\ -r&\left[d_{k+1}\left(\partial_x \Pi_N u(t_{k+1}, x), \partial_x v_N\right) + \sum_{j=0}^{k-1}(b_{j+1} - b_j)d_{k-j}\left(\partial_x \Pi_N u(t_{k-j}, x), \partial_x v_N\right)\right] \\ -r\Upsilon&\left[\left(\Pi_N u(t_{k+1}, x), v_N\right) + \sum_{j=0}^{k-1}(b_{j+1} - b_j)\left(\Pi_N u(t_{k-j}, x), v_N\right)\right] + \tau(g^{k+1}, v_N) + (\tau R_2^{k+1}, v_N). \end{aligned} \tag{5.13}$$

Let $\eta^k = u_N^k - \Pi_N u(t_k, x)$. Then, by using the full discretization (5.2) and (5.13), we obtain that

$$(\eta^{k+1}, v_N) = (\eta^k, v_N) + \tau\left(x\sigma_{k+1}\partial_x\eta^{k+1}, v_N\right) - r\left[d_{k+1}\left(\partial_x\eta^{k+1}, \partial_x v_N\right) + \sum_{j=0}^{k-1}(b_{j+1}-b_j)d_{k-j}\left(\partial_x\eta^{k-j}, \partial_x v_N\right)\right] - r\Upsilon\left[\left(\eta^{k+1}, v_N\right) + \sum_{j=0}^{k-1}(b_{j+1}-b_j)\left(\eta^{k-j}, v_N\right)\right] + (\tau R_2^{k+1}, v_N), \tag{5.14}$$

for any $v_N \in \mathring{P}_N(\Lambda)$. Taking $v_N = \eta^{k+1}$, and performing a similar process to Theorem 4.2, we have

$$Y^{k+1} \le Y^k - rb_k d_{k+1}\|\partial_x\eta^{k+1}\|_0^2 - r\Upsilon b_k\|\eta^{k+1}\|_0^2 - \tau\sigma_{k+1}\|\eta^{k+1}\|_0^2 + 2|(\tau R_2^{k+1}, \eta^{k+1})|, \tag{5.15}$$

where

$$Y^k = \|\eta^k\|_0^2 + r\sum_{j=0}^{k-1} b_j d_{k-j}\|\partial_x\eta^{k-j}\|_0^2 + r\Upsilon\sum_{j=0}^{k-1} b_j\|\eta^{k-j}\|_0^2.$$

Now, $|\tau R_2^{k+1}| \le c_\gamma b_k\tau^{1+\gamma} + C\tau(\tau + N^{2-m})$, thus

$$\begin{aligned} Y^{k+1} &\le Y^k - r\Upsilon b_k\|\eta^{k+1}\|_0^2 - \tau\underline{\theta}\|\eta^{k+1}\|_0^2 + 2|(c_\gamma b_k\tau^{1+\gamma}, \eta^{k+1})| + 2|\left(C\tau(\tau+N^{2-m}), \eta^{k+1}\right)| \\ &\le Y^k - r\Upsilon b_k\|\eta^{k+1}\|_0^2 - \tau\underline{\theta}\|\eta^{k+1}\|_0^2 + \epsilon_1\|\eta^{k+1}\|_0^2 + \frac{1}{\epsilon_1}c_\gamma^2 b_k^2\tau^{2(1+\gamma)} + \epsilon_2\|\eta^{k+1}\|_0^2 + \frac{1}{\epsilon_2}C^2\tau^2(\tau+N^{2-m})^2. \end{aligned} \tag{5.16}$$

Taking $\epsilon_1 = r\Upsilon b_k, \epsilon_2 = \tau\underline{\theta}$, it yields that

$$Y^{k+1} \le Y^k + \frac{b_k\tau^\gamma}{\Upsilon}\Gamma(1+\gamma)c_\gamma^2\tau^2 + \frac{C^2(\tau+N^{2-m})^2}{\underline{\theta}}\tau.$$

Hence,

$$Y^k \le \frac{\Gamma(1+\gamma)c_\gamma^2\tau^2}{\Upsilon}\sum_{j=0}^{k-1} b_j\tau^\gamma + \frac{C^2(\tau+N^{2-m})^2}{\underline{\theta}}\sum_{j=0}^{k-1}\tau,$$

that is, there exists a constant C independent of τ, k and N

$$\|\eta^k\|_0 \le C(\tau + N^{2-m}).$$

Noting that

$$\|u_N^k - u(t_k, x)\|_0 \le \|u_N^k - \Pi_N u(t_k, x)\|_0 + \|u(t_k, x) - \Pi_N u(t_k, x)\|_0,$$

then, the theorem holds. □

6. Implementation

6.1. Galerkin spectral approximation

We firstly consider the solution of the full discrete approximation equation (5.2). Let us take $\varphi_n(x) = L_{n+2}(2x-1) - L_n(2x-1)$, $n = 0, 1, \cdots, N-2$, here $L_n(x)$ denote the Legendre polynomials of degree n. Hence, $\mathring{P}_N(\Lambda) = \text{Span}\{\varphi_0(x), \varphi_1(x), \cdots, \varphi_{N-2}(x)\}$. Let

$$u_N^k(x) = \sum_{j=0}^{N-2} U_j^k\varphi_j(x), \text{ for } k = 0, 1, 2, \cdots, K,$$

and take $v_N = \varphi_l(x), l = 0, 1, \cdots, N-2$, so that from (5.2)

Algorithm 1 Solving the moving boundary problem with homogeneous boundary conditions.

Input: $r, \sigma_i, d_i, \tau, b_i, \Upsilon$.

1. Compute matrices A, B and E;
2. Compute U^0 by solving $BU^0 = c_N$;
3. Compute F_N^1. To solve U^1 by
$$[rd_1A - \tau\sigma_1E + (1 + r\Upsilon)B]U^1 = BU^0 + \tau F_N^1.$$
4. For $k = 1 : K$ do
   - Evaluate F_N^{k+1};
   - Solve the equation (6.1).

Output: $U^0, U^1, \cdots, U^K$. To obtain $u(t_k, x)$.

$$(\partial_x u_N^{k+1}, \partial_x \varphi_l) = \sum_{j=0}^{N-2} (\partial_x \varphi_j, \partial_x \varphi_l)_N U_j^{k+1} = AU^{k+1},$$

$$(u_N^{k+1}, \varphi_l) = \sum_{j=0}^{N-2} (\varphi_j, \varphi_l)_N U_j^{k+1} = BU^{k+1},$$

$$(x\partial_x u_N^{k+1}, \varphi_l) = \sum_{j=0}^{N-2} (x\partial_x \varphi_j, \varphi_l)_N U_j^{k+1} = EU^{k+1},$$

where $(\cdot, \cdot)_N$ denotes the Legendre–Gauss–Lobatto type discrete inner, and

$$A = \left(A_{lj}\right)_{(N-1)\times(N-1)}, B = \left(B_{lj}\right)_{(N-1)\times(N-1)}, E = \left(E_{lj}\right)_{(N-1)\times(N-1)},$$

$$U^{k+1} = (U_0^{k+1}, U_1^{k+1}, \cdots, U_{N-2}^{k+1})^T.$$

Therefore, the full discretization scheme (5.2) can be rewritten in the matrix form as

$$\begin{aligned}&[rd_{k+1}A - \tau\sigma_{k+1}E + (1 + r\Upsilon)B]U^{k+1} \\ &= BU^k + r\sum_{j=0}^{k-1}(b_j - b_{j+1})d_{k-j}AU^{k-j} + r\Upsilon\sum_{j=0}^{k-1}(b_j - b_{j+1})BU^{k-j} + \tau F_N^{k+1},\end{aligned} \tag{6.1}$$

where

$$F_N^{k+1} = \begin{bmatrix} (g^{k+1}, \varphi_0)_N \\ (g^{k+1}, \varphi_1)_N \\ \vdots \\ (g^{k+1}, \varphi_{N-2})_N \end{bmatrix}.$$

At $k = 0$, we have

$$[rd_1A - \tau\sigma_1E + (1 + r\Upsilon)B]U^1 = BU^0 + \tau F_N^1.$$

In order to evaluate U^0, we need to solve the following equation

$$BU^0 = c_N = \begin{bmatrix} (u_0, \varphi_0)_N \\ (u_0, \varphi_1)_N \\ \vdots \\ (u_0, \varphi_{N-2})_N \end{bmatrix}.$$

Since B is symmetric, we may make use of the conjugate gradient (CG) method to solve U^0. For $k \geq 1$, the BiCGSTAB algorithm is adopted to obtain U^k. The algorithm for solving the problem (3.4) is as follows (see Algorithm 1).

6.2. Petrov–Galerkin approximation

In the Galerkin spectral method, the boundary conditions are reduced to homogeneous boundary conditions, the substitutional work is evaluating $f(t,x)$. Now, we consider the Legendre Petrov–Galerkin approximation for Dirichlet boundary conditions. For the moment, we need to consider the boundary conditions, but it is not necessary to compute $f(t,x)$ since $f(t,x)\equiv 0$ here. For this case, the different basis functions shall be considered. Taking $\varphi_n(x)=L_n(2x-1), n=0,1,2,\cdots,N$. Let

$$u_N^k(x)=\sum_{j=0}^{N}U_j^k\varphi_j(x),\ \text{for}\ k=0,1,2,\cdots,K,$$

and take the test functions as $v_j(x)=L_{j+2}(2x-1)-L_j(2x-1)$, $j=0,1,\cdots,N-2$ such that $v_j(x)$ satisfy the homogeneous boundary conditions. Substituting these into (5.2), it gives

$$(\partial_x u_N^{k+1},\partial_x v_l)_N=\sum_{j=0}^{N}(\partial_x\varphi_j,\partial_x v_l)_N U_j^{k+1}=\hat{A}U^{k+1},$$

$$(u_N^{k+1},v_l)_N=\sum_{j=0}^{N}(\varphi_j,v_l)_N U_j^{k+1}=\hat{B}U^{k+1},$$

$$(x\partial_x u_N^{k+1},v_l)_N=\sum_{j=0}^{N}(x\partial_x\varphi_j,v_l)_N U_j^{k+1}=\hat{E}U^{k+1},$$

where $\hat{A},\hat{B},\hat{E}$ are matrices of order $(N-1)\times(N+1)$. Thus, equations (6.1) is changed into ones in which A,B,E are replaced by $\hat{A},\hat{B},\hat{E}$ and F_N^k vanishes, that is

$$\begin{aligned}&[rd_{k+1}\hat{A}-\tau\sigma_{k+1}\hat{E}+(1+r\Upsilon)\hat{B}]U^{k+1}\\&=\hat{B}U^k+r\sum_{j=0}^{k-1}(b_j-b_{j+1})d_{k-j}\hat{A}U^{k-j}+r\Upsilon\sum_{j=0}^{k-1}(b_j-b_{j+1})\hat{B}U^{k-j}.\end{aligned}\tag{6.2}$$

The advantage of the above choice of test functions is that the computation of the first derivative in $\hat{E}$ can be avoided, and that the computation in $\hat{A}$ can become much simpler, by using the properties of Legendre polynomials. In addition, consider the boundary conditions

$$\sum_{j=0}^{N}\varphi_j(0)U_j^{k+1}=u_l^{k+1},\ \text{and}\ \sum_{j=0}^{N}\varphi_j(1)U_j^{k+1}=u_r^{k+1},\tag{6.3}$$

where $u_l^{k+1}=u_l(t_{k+1}), u_r^{k+1}=u_r(t_{k+1})$. Adding the above two equations to (6.2), a closed system is obtained. The algorithm is shown in Algorithm 2.

7. Numerical examples

In this section, we shall study two examples to illustrate the efficiency of our method, and to illustrate our theoretical analysis of the error estimates.

Example 1. Consider the following problem on a non-growing domain:

$$\begin{cases}{}_0^CD_t^{\gamma}u(t,x)=\partial_{xx}u(t,x)-\dfrac{1}{2}u(t,x), 0<x<1, t>0,\\ u(t,0)=u(t,1)=0,\\ u(0,x)=\sin\pi x.\end{cases}\tag{7.1}$$

The simple example serves mainly as testing the theoretical analysis on the error estimate of our method. It is easy to derive the analytical solution of (7.1) as

$$u(t,x)=E_{\gamma,1}\left[-\left(\pi^2+\frac{1}{2}\right)t^{\gamma}\right]\sin\pi x.$$

Algorithm 2 Solving the moving boundary problem with non-homogeneous boundary conditions.

Input: $r, \sigma_i, d_i, \tau, b_i, \Upsilon$.

1. Compute matrices $\hat{A}$, $\hat{B}$, $\hat{E}$ and the corresponding $\tilde{c}_N$;
2. Compute U^0 by solving the system

$$\begin{cases} \hat{B}U^0 = \tilde{c}_N \\ \sum_{j=0}^{N} \varphi_j(0)U_j^0 = u_l^0 \\ \sum_{j=0}^{N} \varphi_j(1)U_j^0 = u_r^0. \end{cases}$$

3. Solve U^1 by

$$[rd_1\hat{A} - \tau\sigma_1\hat{E} + (1 + r\Upsilon)\hat{B}]U^1 = \hat{B}U^0,$$

$$\sum_{j=0}^{N} \varphi_j(0)U_j^1 = u_l^1, \text{ and } \sum_{j=0}^{N} \varphi_j(1)U_j^1 = u_r^1.$$

4. For $k = 1 : K$, solving (6.2) and (6.3).

Output: $U^0, U^1, \cdots, U^K$. To obtain $u(t_k, x)$.

Table 1
L^2-errors of Problem 7.1 versus τ and convergence order, with $N = 13$.

τ	$\gamma = 0.1$		$\gamma = 0.5$		$\gamma = 0.9$	
	L^2-error	Order	L^2-error	Order	L^2-error	Order
1/10	7.7830e–04		8.5949e–03		7.1170e–03	
1/20	4.3978e–04	0.8235	4.2313e–03	1.0224	3.0763e–03	1.2101
1/40	2.4282e–04	0.8569	2.0988e–03	1.0115	1.4358e–03	1.0993
1/80	1.3193e–04	0.8801	1.0458e–03	1.0050	6.9455e–04	1.0477
1/160	7.0827e–05	0.8974	5.2224e–04	1.0018	3.4167e–04	1.0235

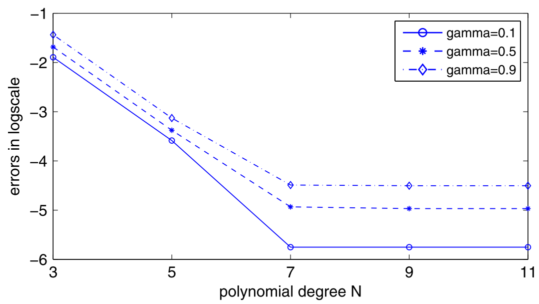

Fig. 1. L^2-errors of the numerical solution of (7.1) in spatial discretization, with $\tau = 5.0e{-}06$.

The numerical results are given in Table 1. Here, we take $N = 13$ which is large enough such that the spatial discretization error is negligible compared with the temporal discretization error. For different fractional derivative orders $\gamma = 0.1, 0.5$ and 0.9, the results in Table 1 show an accuracy $O(\tau)$ is attained. We remark that the analytical solution is not in C^2 near the singular point $t = 0$. As for this type of singularity, McLean and Mustapha proved that the same accuracy can be obtained when nonuniform meshes are used (see [39] for details).

In order to show the spatial discretization error and convergence order, we take $\tau = 5.0 \times 10^{-6}$. The results are plotted in Fig. 1. In the figure, we plot the log-linear error of the numerical solution as a function of polynomial degree N. It can be seen that the errors start decaying exponentially as the error variations are essentially linear versus the degree of polynomial, then stall when they are dominated by temporal discretization errors of order $O(\tau)$. This is also expected by the theoretical results.

Example 2. Consider the moving boundary problem with an exponentially growing domain

$$ {}_0^CD_t^\gamma C(t,x) = \partial_{xx}C(t,x) - C(t,x),\ 0 < x < L(t),\ t > 0, \tag{7.2}$$

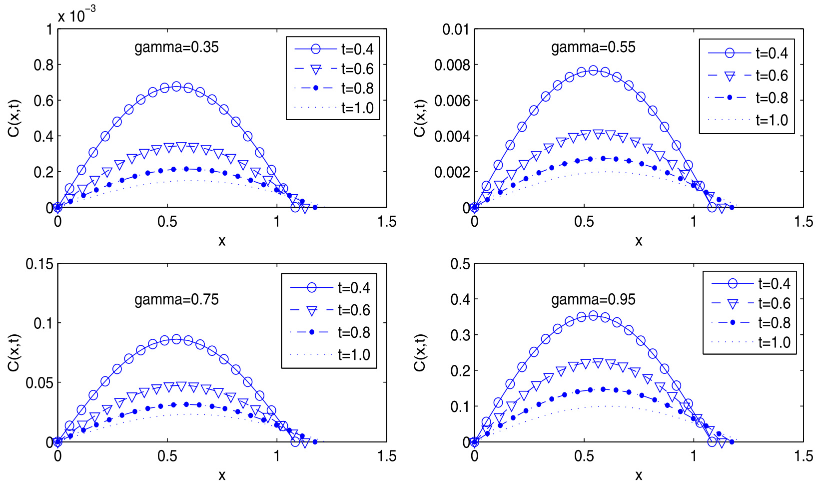

Fig. 2. The solution of moving boundary problem (7.2) with $\gamma = 0.35, 0.55, 0.75, 0.95$ at different times.

where $L(t) = L(0)\exp(0.2t)$. Given the initial condition

$$C(0, x) = \sin \pi x$$

and the Dirichlet boundary conditions $C(t, 0) = 0$, $C(t, L(t)) = 0$.

Let $L(0) = 1$. We check the efficiency of our method for solving some problems on an exponentially growing domain. Taking $\gamma = 0.35, 0.55, 0.75, 0.95$, we see that our method is convergent only if $\tau \leq 0.001$. The results are shown in Fig. 2. These plots show the evolution of $C(t, x)$ and it is clear that the domain is increasing in time. Our results show how the evolution of the moving boundary problem is affected by altering γ.

8. Conclusion

Moving boundary problems are important in many science and engineering applications. However, few studies that examine a fractional diffusion problem with a moving boundary have been presented. In this paper, we first present a fractional reaction–diffusion model with prescribed-boundary moving boundary condition that arises from developmental biology. An efficient numerical method is proposed to solve such a class of fractional moving boundary problem in the present paper. This method utilizes the finite difference scheme to discretize the time variable and a spectral scheme for the space variable. The convergence rate of our method in the temporal discretization is $O(\tau)$ and is spectrally accurate in the spatial discretization. The restriction $\Upsilon > 0$ should be omitted by some technological management, and we shall study it in the future work.

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