Computing residence times for flow towards a pumping well: nomograph solution and validity of the small draw-down approximation

Matthew J. Simpson

Abstract A recent analytical model developed to compute the residence time of fluid flowing in an unconfined aquifer towards a single pumping well is examined. The solution is scaled and presented practically as a nomograph showing the relationship between the residence time, flow length and draw-down. In addition, a similar scaling process is undertaken for the same problem occurring in a confined aquifer so that the error introduced by approximating an unconfined system as a confined system can be understood over a wide range of conditions.

Résumé On analyse un modèle analytique récent pour calculer le temps de résidence d'un fluide pendant son écoulement vers un puits de pompage dans une nappe libre. La solution a été mise-à-l'échelle et présenté d'une manière pratique, comme une nomogramme qui exprime la relation entre le temps de résidence, la distance de l› écoulement et le rabattement. De plus, on a utilisé un procédé similaire de mise-à-l'échelle pour le même problème dans une nappe captive affin que l'erreur introduite par l'approximation d'une nappe libre par une nappe captive peut être interprétée pour une grande classe de conditions.

Resumen Se examina un modelo analítico, recientemente desarrollado, para calcular el tiempo de residencia de un fluido, el cual está fluyendo dentro de un acuífero libre hacia un pozo de bombeo único. La solución después de ser ajustada, se presenta prácticamente como un nomograma, mostrando la relación entre el tiempo de residencia, la longitud del flujo y el abatimiento. Adicionalmente, un proceso similar de ajuste fue realizado para el mismo problema, pero bajo condiciones de acuífero confinado, por tanto el error causado por hacer la aproximación de un sistema libre como si fuera un sistema confinado, puede llegar a ser entendido para un rango amplio de condiciones.

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Introduction

Calculating the time required for the removal of a region of fluid near a pumping well is important when sizing a pump to be used in a pump-and-treat remediation design or for calculating the time taken for a contamination plume to degrade the water quality of a production well. Recently, a new analytical solution to compute the time taken for fluid to be removed by a pumping well under steady-state, unconfined, Dupuit-Forchheimer flow conditions has been developed (Simpson et al. 2003a). This previous work was focused upon deriving the analytical result which involves computing a transcendental function known as the imaginary error function; therefore, the resulting model is not straightforward to implement. The primary aim of this note is to present a more practical discussion where the model is non-dimensionalized and presented as a nomograph.

The development of a non-dimensional solution for the unconfined problem means that the solution can be rigorously compared to the solution for the same problem occurring in a confined aquifer. A widely invoked assumption in analytical groundwater modeling states that an unconfined problem can be approximated as a confined problem where the draw-down (variation in saturated thickness) is small. Unconfined flow problems are inherently non-linear and so the availability of analytical solutions is limited. Therefore, for this particular problem it has been impossible to quantify how small the draw-down has to be such that the unconfined equations can be approximated by the confined equations within a known error boundary. Comparing the non-dimensional solutions for confined and unconfined flow conditions quantifies how the systems diverge under larger draw-down conditions.

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Results: Non-dimensionalization

Simpson et al. (2003a) showed that the time taken for the removal of fluid near a well is given by:

$$t_{uc} = -\frac{2\theta \log_e (r_R/r_W)}{K\left(h_R^2 - h_W^2\right)} \int_{r_R}^{r_W} \cdot r\sqrt{h_w^2 + \frac{\left(h_R^2 - h_W^2\right)}{\log_e (r_R/r_W)} \log_e\left(r/r_w\right)}\,dr, \quad (1)$$

where t_{uc} is the time taken for the fluid in an unconfined aquifer to move from r_R (L), the radius of influence, to r_W (L), the radius of the well casing, θ is the porosity of the porous medium, K (LT^{-1}) is the hydraulic conductivity of the porous medium, r (L) is the radial coordinate, h_R (L) is the hydraulic head at the radius of influence and h_W (L) is the hydraulic head at the well casing. The physical layout of this problem along with the boundary conditions is shown in Fig. 1a.

The integral in Eq. (1) can be solved by using integration by parts and an appropriate substitution to yield the solution:

$$t_{uc} = -\frac{2\theta \log_e (r_R/r_W)}{K\left(h_R^2 - h_W^2\right)} \left| \frac{r^2}{2}\sqrt{h_w^2 + \frac{\left(h_R^2 - h_W^2\right)}{\log_e (r_R/r_W)} \log_e\left(r/r_w\right)} - \left[\frac{r_W^2}{4}\sqrt{\frac{\pi\left(h_R^2 - h_W^2\right)}{2\log_e (r_R/r_W)}} \exp\left[-\frac{2h_W^2 \log_e (r_R/r_W)}{\left(h_R^2 - h_W^2\right)}\right] \times \mathrm{Erfi}\left\{\sqrt{\frac{2\log_e (r_R/r_W)}{\left(h_R^2 - h_W^2\right)}} \cdot \sqrt{h_W^2 + \frac{\left(h_R^2 - h_W^2\right)}{\log_e (r_R/r_W)} \log_e(r/r_W)}\right\}\right]\right|_{r_R}^{r_W} \quad (2)$$

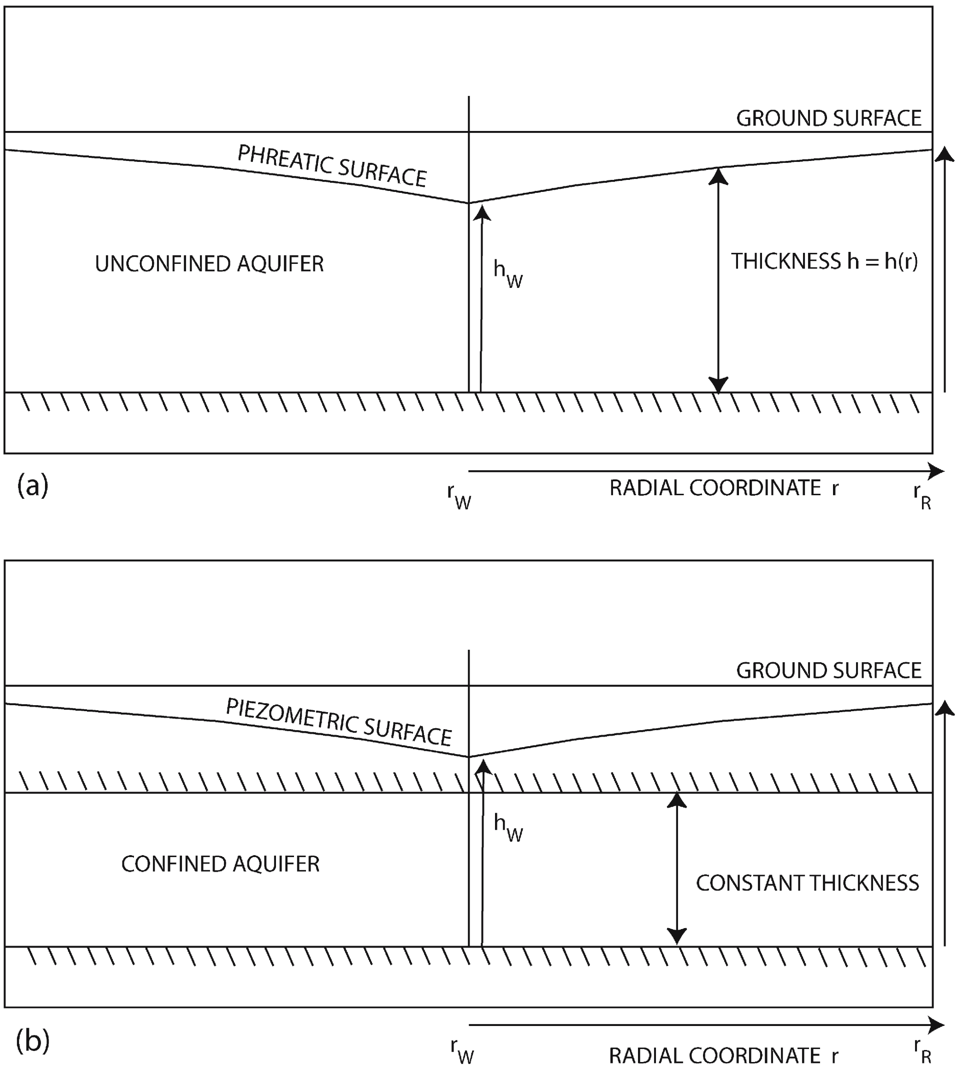

Fig. 1 Conceptual diagrams showing flow near a well under **a** unconfined conditions, and **b** confined conditions

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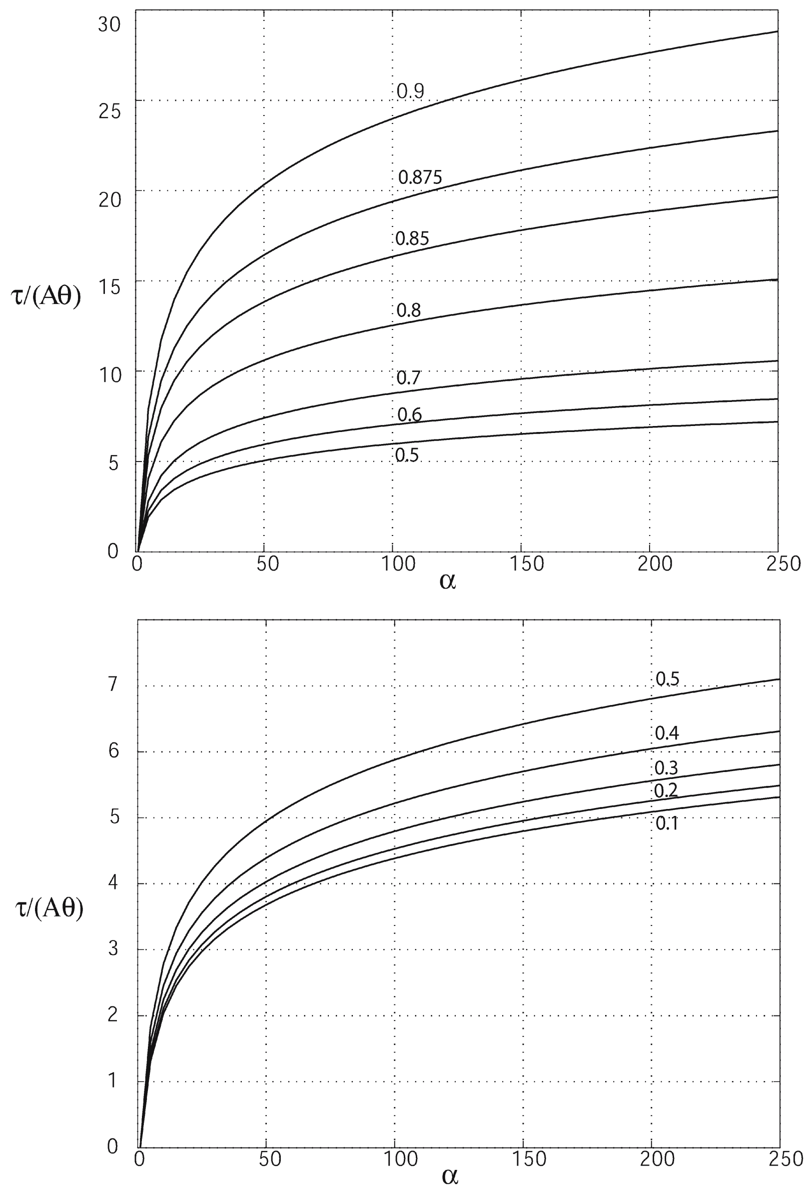

Fig. 2 Nomograph showing non-dimensional residence time as a function of the non-dimensional flow length and non-dimensional draw-down (values of β are indicated)

where Erfi (x) is the imaginary error function, given by $\mathrm{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(u^2)\mathrm{du}$. The imaginary error function can be conveniently evaluated using a Maclaurin series expansion (Simpson et al. 2003a).

To non-dimensionalize the expression in Eq. (2), the following transformations are used and new non-dimensional parameters A, α, and β are introduced:

$$h^* = \frac{h}{\mathrm{h_R}}, \quad r^* = \frac{r}{\mathrm{r_R}}, \quad \tau^* = \frac{t\mathrm{K}}{\mathrm{r_R}}, \tag{3}$$

$$\mathrm{A} = \frac{\mathrm{r_R}}{\mathrm{h_R}}, \quad \alpha = \frac{\mathrm{r_R}}{\mathrm{r_W}}, \quad \beta = \frac{\mathrm{h_W}}{\mathrm{h_R}}. \tag{4}$$

Note that A can be thought of as an aspect ratio for the problem, α is a non-dimensional radial flow length, β is a non-dimensional draw-down and τ is a non-dimensional time related to the real time taken for fluid moving at a velocity K to traverse a distance of $\mathrm{r_R}$.

Using these definitions and neglecting the asterisk notation, an equivalent non-dimensional integral expression for the unconfined residence time is:

$$\tau_{\mathrm{uc}} = -\frac{2\mathrm{A}\theta \log_{\mathrm{e}}(\alpha)}{1-\beta^2} \int_1^{\frac{1}{\alpha}} r\sqrt{\beta^2 + \frac{1-\beta^2}{\log_{\mathrm{e}}(\alpha)} \log_{\mathrm{e}}(\mathrm{r}\alpha)}\,\mathrm{dr}, \tag{5}$$

where τ_{uc} is the non-dimensional residence time under unconfined flow conditions. Following a similar integration by parts and substitution, the non-dimensional solution can be written as:

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$$\tau_{uc} = -\frac{2A\theta\log_e(\alpha)}{1-\beta^2}\left[\frac{\beta-\alpha^2}{2\alpha^2} + \frac{1}{4\alpha^2}\sqrt{\frac{\pi(1-\beta^2)}{2\log_e(\alpha)}}\exp\left\{\frac{-2\beta^2\log_e(\alpha)}{1-\beta^2}\right\} \times\left(\mathrm{Erfi}\left\{\sqrt{\frac{2\log_e(\alpha)}{1-\beta^2}}\right\} - \mathrm{Erfi}\left\{\beta\sqrt{\frac{2\log_e(\alpha)}{1-\beta^2}}\right\}\right)\right]. \tag{6}$$

Equation (6) is a non-dimensional solution for the time taken to remove a region of fluid around a pumping well in an unconfined aquifer. The solution still depends upon computing the imaginary error function; however, in this more general form, the solution can be presented graphically to show the non-dimensional residence time as a function of non-dimensional flow length and draw-down.

The non-dimensional solution is presented in Fig. 2 where the vertical axis relates to the non-dimensional residence time and the horizontal axis represents the radial flow length. The values are plotted over a range of $0.1\leq\beta\leq0.9$ and $1.0\leq\alpha\leq250.0$.

Figure 2 shows that the residence time increases as both α and β increase. The presentation of the results in the nomograph form is very useful since the solution can be simply read, or interpolated, from the graph without having to explicitly compute the imaginary error function; and, the figure is applicable over a wide range of boundary conditions. If, for example, the residence time was to be computed where r_W=0.1 m, r_R=10.0 m, h_W=3.5 m, h_R=4.0 m, K=50.0 m/day and θ=0.3, then the non-dimensional parameters, Eq. (4), are A=2.5, α=100 and β=0.875. The non-dimensional time is simply read from Fig. 2 giving the residence time t =2.9 days, which agrees with the dimensional calculation performed by Simpson et al. (2003a), but without the need to explicitly compute the imaginary error function.

Confined and Unconfined Flow: How Small is Small?

Unconfined Dupuit-Forchheimer flow is non-linear and so analytical solutions describing these conditions are less readily available than for linear confined flow. Therefore, often in practice it is typical to invoke the small draw-down assumption where an unconfined aquifer is assumed to behave like a confined aquifer for a small draw-down. This classical strategy is very common and quite useful since the non-linear problem is readily linearized (e.g. Bear 1972; Freeze and Cherry 1979; Haitjema 1995; Muskat 1937; Theis 1935).

Now that the new analytical solution for unconfined flow has been generalized into a non-dimensional form, it is convenient to compare the solution for an equivalent problem occurring in a confined aquifer. This comparison is useful to show the conditions where the small draw-down approximation are valid. Using the same notation as for the unconfined case, the residence time in a confined aquifer with a constant thickness (Fig. 1b) is:

$$t_c = -\frac{\theta\log_e(r_R/r_W)}{K(h_R-h_W)}\int_{r_R}^{r_W} r\,dr, = -\frac{\theta\log_e(r_R/r_W)}{2K(h_R-h_W)}\left(r_W^2-r_R^2\right). \tag{7}$$

Unlike the unconfined case, Eq. (1), the evaluation of the expression for residence time in a confined aquifer, Eq. (7), is straightforward. The expressions for confined flow can be non-dimensionalized using identical scalings and parameters for the unconfined case to yield an equivalent non-dimensional expression for the residence time of fluid flowing in a confined aquifer:

$$\tau_c = -\frac{A\theta\log_e(\alpha)(1-\alpha^2)}{2(1-\beta)\alpha^2}, \tag{8}$$

where τ_c is the non-dimensional residence time in a confined aquifer.

The mathematical expressions for the non-dimensional residence time under confined, Eq. (8), and unconfined, Eq. (6), conditions are strikingly different! This difference can be explained physically by considering the influence of the differences shown in Fig. 1. In order to compare confined and unconfined conditions a fixed set of aquifer parameters and boundary conditions must be considered. For a confined problem, as the fluid moves towards the pumping well the fluid velocity increases as the area normal to the flow decreases due to the radial geometry. Under unconfined conditions, the fluid velocity also increases toward the pumping well due to a reduction in the area normal to the flow; however, there are two mechanisms contributing to the reduced area. Firstly, the radial geometry contributes to a reduced area normal to the flow as for confined conditions. Secondly, the radial variation of the phreatic surface elevation causes a vertical reduction in the area normal to the flow. This vertical variation in the saturated thickness gives rise to the divergence between the confined and unconfined models. Given that the only difference between these models is caused by the variation in saturated thickness, it is reasonable to assume that the unconfined problem will behave similarly to a confined problem for a small draw-down. The disadvantage of relying solely upon this physical argument is that the divergence between the two models cannot be physically quantified and no insight into how the error grows with increasing draw-down is possible. To overcome this limitation, a mathematical analysis is preferred.

To mathematically consider the divergence of the confined and unconfined models it is instructive to consider the relationship between the non-dimensional times τ_{uc} and τ_c. As previously stated, the unconfined solution should relax to the confined solution when the draw-down approaches zero ($\beta\rightarrow1.0$), this can be confirmed using a measure of the absolute difference between the two models, $\mathrm{Error}_{absolute}=|\tau_{uc}-\tau_c|$. As expected, the absolute error equals zero in the limit $\beta\rightarrow1.0$.

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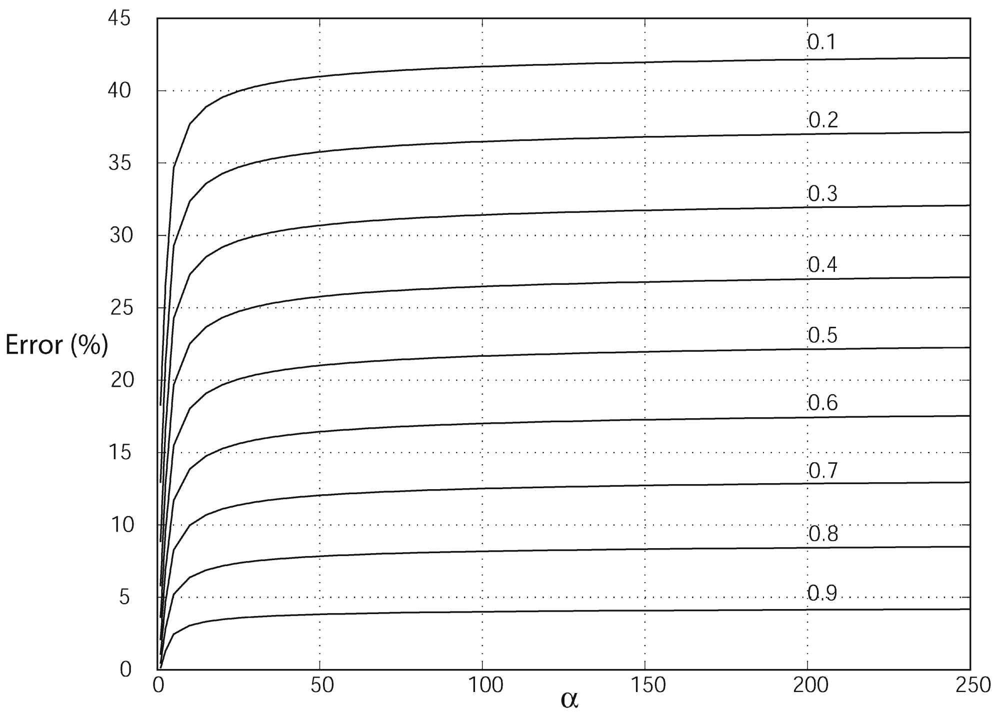

Fig. 3 Relative error analysis showing the divergence between the confined and unconfined models in the range, $0.1 \leq \beta \leq 0.9$ and $1.0 \leq \alpha \leq 250.0$ (values of β are indicated)

It is more interesting and informative to consider the difference between the confined and unconfined models for intermediate values of β. For this purpose, a relative error is defined:

$$\mathrm{Error}_{\mathrm{relative}}(\%) = 100 \times \frac{|\tau_{\mathrm{uc}} - \tau_{\mathrm{c}}|}{\tau_{\mathrm{uc}}}. \quad (9)$$

The relative error (9) enables the quantification of how the confined and unconfined models diverge. For instance, the previous numerical example with the draw-down of β=0.875, over a flow length of α=100, yields residence times of τ_{uc}=14.55 and τ_c=13.81: the small draw-down error is 5.1% according to Eq. (9). If we consider the case where the draw-down is increased, β=0.5, it is not immediately clear how sensitive the small draw-down approximation error will be to this increase in draw-down. Evaluating the expressions for the increased draw-down gives τ_{uc}=4.41 and τ_c=3.45 with an error of 21.8%. The increase in the small draw-down error is a result of the increased variation in the saturated thickness.

It is interesting to note that the relative error, Eq. (9), is independent of the aquifer parameters K and θ. The error expression Eq. (9) depends only upon the geometry of the problem defined through the boundary conditions. It follows that any analysis of the small draw-down error, defined by Eq. (9) with the models Eqs. (6) and (8), requires no knowledge or measurement of the aquifer properties K and θ.

In order to explore the relationship between the confined and unconfined models over a wide range of conditions, the value of the relative error Eq. (9) has been computed over the range $1.0 \leq \alpha \leq 250.0$ and $0.1 \leq \beta \leq 0.9$, and plotted in Fig. 3. The graphical representation of the relative error (Fig. 3) shows two trends. First, for a given draw-down β, the error increases with increasing distance α. However, at some point the error no longer increases with α, this occurs as the saturated thickness does not vary significantly at large distances from the pumping well and so the system behaves as a confined system. There appears to be a critical value of $\alpha \approx 50$, where beyond this point the unconfined system is approximated very well by the confined model; conversely, in the region closer to the pumping well the two models diverge rapidly since the draw-down is significant. It is impossible to quantify these regimes without the non-dimensional solution. The second trend in the error analysis shows how the errors increase as the draw-down increases. Although this trend is physically intuitive, it is not possible to quantify the divergence between the confined and unconfined models over a broad parameter range without the non-dimensional analysis presented here.

The trends in the error analysis show clear asymptotic behavior for large α (Fig. 3). It is useful to formalize this trend since for each value of β there exists an upper boundary to the error between the two models which occurs in the limit $\alpha \to \infty$. Evaluating the relative error expression Eq. (9) under these conditions yields:

$$\mathrm{Error}_{\alpha \to \infty}(\%) = 50(1 - \beta). \quad (10)$$

The asymptotic error trend, Eq. (10) gives an upper boundary to the error between the confined and unconfined models. This simple expression accurately predicts behavior of the error for large flow lengths and is easily verified with the results in Fig. 3. Finally, it should be noted that the relative error (9) is undefined when α=1.0.

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In preparing Fig. 3, this complication was circumvented by expanding the error expression (9) asymptotically in the limit $\alpha \rightarrow 1.0$. Interestingly, this analysis showed that the error is non-zero in the limit $\alpha \rightarrow 1.0$. A thorough discussion of the behavior of the error analysis under these conditions is unwarranted here because this limit has little practical significance.

Finally, it is worthwhile to acknowledge the limitations of the assumptions behind the models presented in this paper. Dupuit-Forchheimer unconfined flow models are known to accurately represent unconfined groundwater flow conditions where the horizontal length scale dominates (Haitjema 1995). However, Dupuit-Forchheimer models are not reliable where vertical processes dominate as Dupuit-Forchheimer models are unable to capture vertical flow components and seepage-face dynamics at the point of outflow (e.g. Clement et al. 1996; Simpson et al. 2003b). Vertical flow processes dominate under conditions where the residence time is evaluated over small horizontal flow lengths (small α) and the draw-down is large (small β). While this cautionary observation is important, it does not lessen the significance of the results presented here since Dupuit-Forchheimer models are known to perform adequately for many field-scale applications (Haitjema 1995). In addition, further limitations of the unconfined residence time model of Eq. (2) are discussed in Simpson et al. (2003a).

Conclusion

A recent analytical model describing the residence time of fluid near a pumping well in an unconfined aquifer was considered and presented in a convenient non-dimensional form. The non-dimensional solution is presented as a nomograph. This enables the model to be evaluated over a wide range of conditions without the need to explicitly compute the imaginary error function.

In addition to presenting the non-dimensional results, the unconfined solution is rigorously compared to a non-dimensionalized solution for an identical problem occurring in a confined aquifer. The confined and unconfined models are compared in order to quantify how the two solutions diverge as the draw-down increases. This analysis exactly identifies how small the draw-down has to be so that the confined flow equations can be used to mimic unconfined conditions within a known error boundary.

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