

# Analytical Model for Computing Residence Times Near a Pumping Well

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## Abstract/

An analytical solution for calculating the residence time of fluid flowing toward a pumping well in an unconfined aquifer has been developed. The analytical solution was derived based on a radial, steady-state, Dupuit-Forchheimer flow model. The resulting integral expression involved computing the imaginary error function, for which a simple series expansion is proposed. The validity of the analytical expression is demonstrated by testing its results against numerical results for an example problem. The analytical solution compared favorably with the numerical approximation.

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## Introduction

Calculating the time required for the removal of a region of fluid near a pumping well is of interest when designing a pump-and-treat system for contaminant removal or when determining the time a contaminant plume may impact the water quality of a pumping well. In this study, we focus on developing an analytical model for computing the residence time for the flow near a well in an unconfined aquifer. The use of analytical models for describing ground water flow near a pumping well has been previously documented, but the focus in these works has been on developing expressions for confined flow (Bear 1979; Javandel and Tsang 1986). Some investigations have analyzed unconfined flow, but the focus was to delineate capture zones when the well flow was superimposed on a background regional flow (Strack 1989; Grubb 1993).

The objective of this work is to develop an analytical model for calculating the residence time of ground water removed through a pumping well in an unconfined aquifer. The Dupuit-Forchheimer model is used to describe the unconfined fluid flow in radial coordinates about a well operating under steady-state conditions. An analytical method for the solution of the governing flow equation is

presented. The analytical solution is validated against a numerical solution. One of the limitations of the proposed analytical method is that the method ignores the influence of regional ground water flow; therefore, it should not be used for situations where there is a strong regional flow.

## Problem Description

Consider ground water flowing under steady-state conditions through an unconfined aquifer. The flow is directed toward a single pumping well, and the aquifer is bounded from below by an impermeable unit. The upstream boundary consists of a constant head boundary condition where the hydraulic head is maintained at  $h_R$  [L] at the radius of influence  $r_R$  [L] from the center of the pumping well. The downstream boundary also consists of a constant head condition where the phreatic surface has been lowered to correspond with a hydraulic head of  $h_W$  [L] at the well casing,  $r_W$  [L].

The Dupuit-Forchheimer model (commonly known as the Dupuit model) is used to describe the flow around the pumping well. The original work of Dupuit (1863) proposed that the total extraction rate from a pumping well could be related to the drawdown of the phreatic surface about the well (refer to Bear [1979] for concept figures and derivations):

$$Q = \pi K \frac{(h_R^2 - h_W^2)}{\log_e (r_R/r_W)} \quad (1)$$

where  $Q$  [ $L^3T^{-1}$ ] is the total extraction rate,  $h_W$  [L] is the hydraulic head at the well casing,  $h_R$  [L] is the hydraulic head at the radius of influence, and  $r_W$  [L] and  $r_R$  [L] are

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the radial distance to the well casing and the radius of influence, respectively.

To describe the distribution of the phreatic surface around the pumping well, Darcy's law is introduced into Equation 1, and the expression is integrated to yield the distribution of the fluid elevation as a function of the radial distance from the center of the pumping well (Bear 1979):

$$h(r) = \sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)} \quad (2)$$

Because we are interested in the fluid residence time, the fluid velocities within the aquifer must be determined; this is achieved by using Darcy's law in the form:

$$v(r) = -\frac{K}{\theta} \frac{dh(r)}{dr} \quad (3)$$

where  $v(r)$  [ $LT^{-1}$ ] is the fluid velocity,  $K$  [ $LT^{-1}$ ] is the hydraulic conductivity of the porous medium,  $\theta$  is the porosity of the porous medium, and  $dh/dr$  is the hydraulic gradient.

Equations 1 through 3 describe the steady-state fluid flow around a single pumping well in an unconfined aquifer. Because we need to determine the time required for a region of fluid to be removed from the aquifer, we must first describe the velocity of the fluid along the radial coordinate system, as it flows toward the pumping well, which can be represented as

$$\frac{dr}{dt} = v(r) \quad (4)$$

Equation 4 equates the rate of change of position of the fluid to the fluid velocity at a particular point in the flow field. Integrating this differential equation yields an expression for the travel time between two arbitrary points  $r_1$  and  $r_2$  in the velocity field:

$$t = -\frac{2\theta \log_e(r_R/r_w)}{K(h_R^2 - h_w^2)} \int_{r_1}^{r_2} \sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)} dr \quad (5)$$

where  $r_1$  [L] is the release point in the flow field and  $r_2$  [L] is the location of the fluid after a time  $t$  [T]. The objective now is to develop an analytical expressions for the definite integral (5).

## Evaluating the Residence Time

### Analytical Method

Before evaluating the integral expression (Equation 5), it should be noted as a caveat that the following analysis is undertaken under the assumption that the argument of the

square root term in Equation 5 is always positive. We can be guaranteed that this term will always be positive, because it is identical to the term on the right-hand side of Equation 2. This term represents the distribution of the hydraulic head within the aquifer; therefore, we can be guaranteed that when using physically meaningful constants, the issue of a complex solution would never arise.

The integral expression (Equation 5) can be simplified by the application of integration by parts:

$$t = -\frac{2\theta \log_e(r_R/r_w)}{K(h_R^2 - h_w^2)} \left( \frac{r^2}{2} \sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)} - \frac{(h_R^2 - h_w^2)}{4 \log_e(r_R/r_w)} \int_{r_1}^{r_2} \frac{r}{\sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)}} dr \right) \quad (6)$$

The integral appearing in Equation 6 can then be evaluated by using the transformation

$$u = \sqrt{\frac{2 \log_e(r_R/r_w)}{(h_R^2 - h_w^2)}} \sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)} \quad (7)$$

After applying the transformation (Equation 7), the integral expression in Equation 6 can be simplified:

$$\int_{r_1}^{r_2} \frac{r}{\sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)}} dr = \sqrt{\frac{2 \log_e(r_R/r_w)}{(h_R^2 - h_w^2)}} \int_{u_1}^{u_2} r^2 du \quad (8)$$

where  $u_1$  and  $u_2$  are the transformed limits of integration, obtained by applying the transformation (Equation 7) to the original limits,  $r_1$  and  $r_2$ .

Integrating Equation 8 and employing the definition of the imaginary error function,  $\text{Erfi}(x)$  yields the solution:

$$t = -\frac{2\theta \log_e(r_R/r_w)}{K(h_R^2 - h_w^2)} \left[ \frac{r^2}{2} \sqrt{h_w^2 + \frac{(h_R^2 - h_w^2)}{\log_e(r_R/r_w)} \log_e(r/r_w)} \right.$$

$$\left. - \left[ \frac{r_w^2}{4} \sqrt{\frac{\pi(h_R^2 - h_w^2)}{2 \log_e(r_R/r_w)}} \exp \left\{ -\frac{2h_w^2 \log_e(r_R/r_w)}{(h_R^2 - h_w^2)} \right\} \right] \times \right.$$

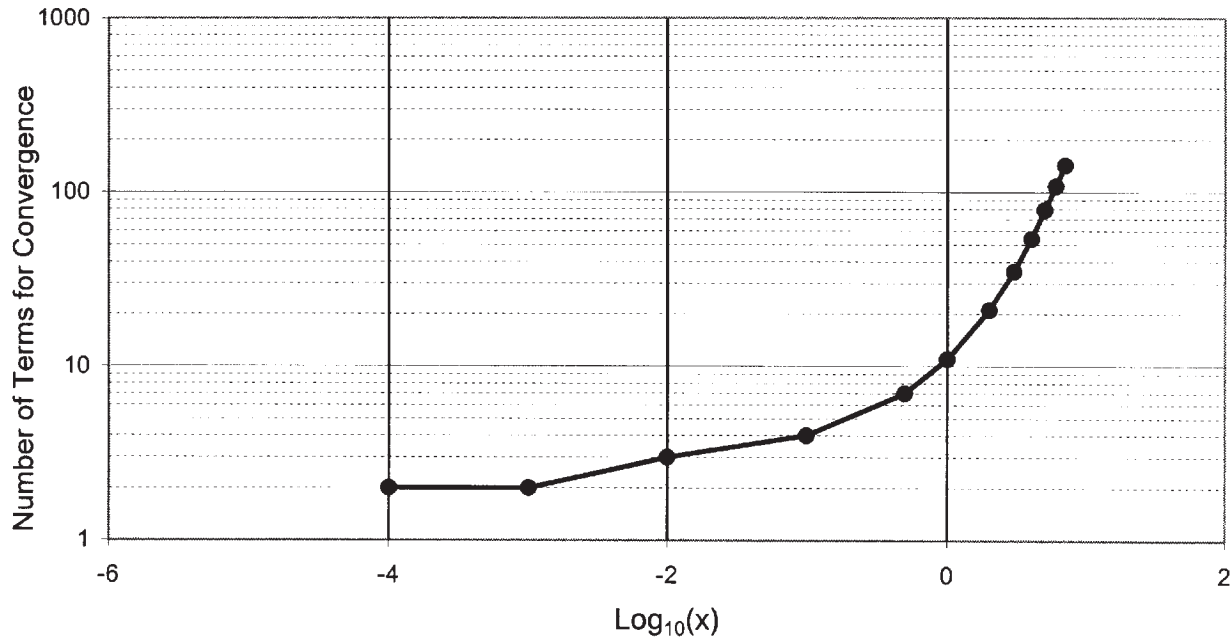


Figure 1. Convergence behavior of the Maclaurin series expansion of the imaginary error function for  $\epsilon = 1 \times 10^{-7}$ .

$$\text{Erfi} \left( \sqrt{\frac{2 \log_e (r_R / r_W)}{(h_R^2 - h_W^2)} \sqrt{h_W^2 + \frac{(h_R^2 - h_W^2)}{\log_e (r_R / r_W)} \log_e (r / r_W)}} \right) \Bigg|_{r_1}^{r_2} \quad (9)$$

where  $\text{Erfi}(x)$  is the imaginary error function of  $x$ , defined by (Wolfram 1996)

$$\text{Erfi}(x) = \frac{2}{\sqrt{\pi}} \int_0^x \exp(u^2) du \quad (10)$$

The new expression (Equation 9) is an analytical approach for calculating the residence times for fluid flow toward a pumping well in an unconfined aquifer.

#### Evaluation of the Imaginary Error Function

Before the analytical expression (Equation 9) can be evaluated, it is necessary to calculate the imaginary error function (Equation 10). The imaginary error function is a standard function known in the theory of heat conduction (Carslaw and Jaeger 1959). Because the function to be integrated is an exponential function with its argument squared, it is a rapidly increasing function. The nature of this function makes the evaluation of Equation 10 more complicated than other commonly used functions, such as the standard error function. Note that the standard error function has the same form as Equation 10, but the exponential is negative; therefore, the function is bounded. The standard error function gradually varies between  $\text{erf}(0) = 0$  and  $\text{erf}(\infty) = 1$ , which makes tabulated values of the error function an attractive option for its evaluation. Because the imaginary error function is unbounded and rapidly increasing, tabulation is not a good option. Some attempts, however, have been made in the past to compile approximate values of the imaginary error function. For exam-

ple, Miller and Gordon (1931) proposed that the integral be re-expressed in terms of two new sinusoidal functions, which would allow the evaluation of the integral through the use of two double-entry tables.

In this work, we derived the following Maclaurin series expansion for the imaginary error function:

$$\begin{aligned} \text{Erfi}(x) &= \frac{2}{\sqrt{\pi}} \left[ \frac{x^1}{1 \times 0!} + \frac{x^3}{3 \times 1!} + \frac{x^5}{5 \times 2!} + \frac{x^7}{7 \times 3!} + \frac{x^9}{9 \times 4!} + \dots \right] \\ &= \frac{2}{\sqrt{\pi}} \sum_{i=1}^{\infty} \frac{x^{(2i-1)}}{[2i-1][(i-1)!]} \end{aligned} \quad (11)$$

Equation 11 is a convenient method for the evaluation of the imaginary error function. Applying the ratio test to the successive terms in Equation 11 indicated that the series converges for any argument. Therefore, realistically, the series can be truncated after a finite number of terms. To aid in the practical evaluation of the series (Equation 11), it is possible to relate the argument of the series to the number of terms 'n' in the truncated series, such that the magnitude of the 'n+1' term falls below some specified tolerance. For example, Figure 1 shows a relationship between the argument  $x$  and the number of terms in Equation 11 required such that the magnitude of the first truncated term is less than  $\epsilon = 1 \times 10^{-7}$ . The figure shows that, as the argument of the imaginary error function increases, the number of terms required in the series also increases. Figure 1 can be used as a practical guide for evaluating the required number terms for computing Equation 11.

#### Numerical Solution

To evaluate the travel time numerically, we applied the trapezoidal algorithm to directly solve Equation 5 (Press et al. 1992). The algorithm was applied using an increasing

number of uniform trapezoids until the final result converged. This converged solution was taken as the numerical approximation for the travel time and was used to compare with the proposed analytical result. Note that the numerical integration of Equation 5 is indeed straightforward and less tedious to compute than the proposed analytical Equation 9. This should not, however, lessen the importance of the analytical expression, which is an exact solution to the problem.

### Example Calculation

The analytical expression, Equation 9, was tested against a numerical approximation to the residence time for a field-scale problem. The problem consists of a 5 m deep unconfined stratum, characterized by  $K = 50$  m/day and  $\theta = 0.3$ . The original phreatic surface was 4 m above the base of the aquifer, and the steady drawdown resulted in a saturated thickness of 3.5 m at the well casing. The drawdown is caused by pumping at a rate of 28 m<sup>3</sup>/day through a pumping well of radius 0.1 m. If the radius of influence of the pumping is 10 m from the center of the pumping well, then we can calculate the residence time required for the removal of fluid within a particular distance from the pumping well. For this example, the residence times for fluid to travel from a distance of  $r = 10$ ,  $r = 5$ , and  $r = 2$  m to the casing where  $r = 0.1$  m was calculated. The application of Equation 9 indicates that the time of removal was 2.909 days from 10 m, 0.714 day from 5 m, and 0.111 day from 2 m. Numerical integrating Equation 5 yielded results identical to those predicted by the analytical solution.

### Conclusions

An analytical solution for calculating the residence time of fluid in an unconfined aquifer subject to a single pumping well has been developed. The evaluation of the analytical expression involved the calculation of the imaginary error function. A simple series expansion for this function was proposed. The analytical formulation was tested for a theoretical problem where the analytical residence time was checked against a numerical approximation; the results were identical.

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