Applied Mathematics Letters 132 (2022) 108151

Contents lists available at ScienceDirect

Applied Mathematics Letters

www.elsevier.com/locate/aml

Exact time-dependent solutions of a Fisher–KPP-like equation obtained with nonclassical symmetry analysis

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ARTICLE INFO

Article history:
Received 24 February 2022
Received in revised form 23 April 2022
Accepted 23 April 2022
Available online 6 May 2022

Keywords:
Fisher–Kolmogorov
Nonclassical symmetry analysis
Reaction–diffusion
Population biology

ABSTRACT

We consider a family of exact solutions to a nonlinear reaction–diffusion model, constructed using nonclassical symmetry analysis. In a particular limit, the mathematical model approaches the well-known Fisher–KPP model, which means that it is related to various applications including cancer progression, wound healing and ecological invasion. The exact solution is mathematically interesting since exact solutions of the Fisher–KPP model are rare, and often restricted to long-time travelling wave solutions for special values of the travelling wave speed.

1. Introduction

Reaction–diffusion models are commonplace in applied mathematics. These involve one or more parabolic partial differential equations (pdes) that, in one dimension, are written as

$$\frac{\partial u}{\partial t} = \frac{\partial}{\partial x}\left(D(u)\frac{\partial u}{\partial x}\right) + R(u). \tag{1}$$

We discuss possible forms for $D(u)$ and $R(u)$ shortly, but note for the purposes of our study, that one of the most popular examples of (1) is the Fisher–KPP equation (in dimensionless form)

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \tag{2}$$

[1,2], which is used extensively. The overall theme of our work is to study a family of exact solutions of a version of (1) that is closely related to solutions of (2).

More generally, in reaction–diffusion models of the form (1), the nonlinear reaction term $R(u)$ can be used to model reproduction in ecology [3,4] or proliferation in cell biology [5,6]. A common form for this reaction

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https://doi.org/10.1016/j.aml.2022.108151
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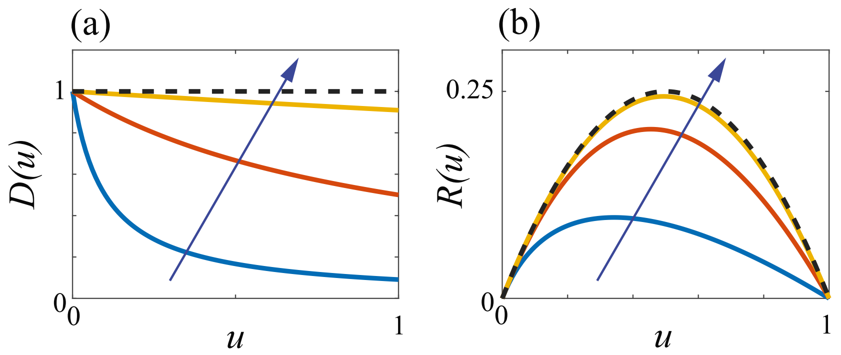

Fig. 1. Nonlinear diffusion $D(u)$ **and source term** $R(u)$. Functions defined in (4) are shown for $\beta = 0.1$ (blue), $\beta = 1$ (orange) and $\beta = 10$ (yellow), with the arrows indicating the direction of increasing β. Also included as a dashed black curve is the constant diffusivity and logistic growth functions from the Fisher–KPP Eq. (2), which $D(u)$ and $R(u)$ approach in the limit $\beta \to \infty$.

term is (in dimensional variables) the logistic growth term $R(u) = \lambda u(1-u/K)$, where λ is the reproduction rate and K is the carrying capacity. Indeed, this is the reaction term in (2). A feature of the logistic term is that the per capita growth rate $\lambda(1 - u/K)$ linearly decreases to zero at $u = K$. Other qualitatively similar monostable reaction terms $R(u)$ that retain key properties, including a single local maximum and $R(0) = R(K) = 0$, are routinely studied in the literature [4,7]. In this sense, logistic growth falls into a broader, well-studied class of qualitatively similar reaction terms.

Forms for the nonlinear diffusivity $D(u)$ in (1) vary depending on the application, although linear diffusion is the most commonly used form (as in the Fisher–KPP model (2)). One option is to take the nonlinear diffusion to be $D(u) = u^n$, where $n > 0$ [6,8]. Of particular relevance to our study, in the context of modelling biological cells, is a nonlinear diffusivity $D(u)$ that is a decreasing function of u [9,10].

Much attention has been devoted to deriving exact solutions to versions of (1). Travelling wave solutions of the Fisher–KPP Eq. (2) were first presented by Ablowitz & Zepatella [11] and then others [12,13]. For the more general class of reaction–diffusion Eqs. (1) with linear diffusion, a number of travelling wave solutions exist [14–16] and some periodic solutions have been found [17,18]. Nonclassical symmetry solutions have been constructed [19,20], and these solutions can also be found using Painlevé analysis [21,22]. Fewer exact analytical solutions exist when the diffusivity is nonconstant (and the reaction term is nonzero); however, some nonclassical symmetry solutions for particular forms of $D(u)$ and $R(u)$ have been presented in Refs. [23–27].

Here we report on a family of exact analytical solutions to (1),

$$u(x,t) = \beta \left[\exp\left(\ln\left(1 + \frac{u_0}{\beta}\right) \exp(-\beta t) \cos\left(\sqrt{1+\beta}\, x\right)\right) - 1 \right], \quad 0 < x < \frac{\pi}{2\sqrt{1+\beta}}, \tag{3}$$

which hold for

$$D(u) = \frac{\beta}{u+\beta}, \quad R(u) = \beta(1-u)\ln\left(1 + \frac{u}{\beta}\right). \tag{4}$$

The form of $R(u)$ and $D(u)$ in (4) is such that (1) is analogous to (2), as indicated in Fig. 1. For example, $R(u)$ is a monostable reaction term with zeros at $u = 0$ and 1, just like the logistic growth term in (2). Furthermore, the one-parameter family $R(u)$ approaches the logistic growth term in the limit $\beta \to \infty$, as is clear from Fig. 1(b). The nonlinear diffusivity $D(u)$ in (4) is a positive decreasing function of u for all $u > 0$. Again, the family of functions $D(u)$ approaches the constant $D(u) \equiv 1$ in the limit $\beta \to \infty$, which reinforces the analogy between these models. Therefore, the exact solutions (3) fall into the same class of solutions to (2) and we discuss this connection in some detail.

Very briefly, the property that $D(u)$ is a decreasing function of u can be used to model density-dependent cell motility. Under some circumstances, cells that are more crowded may be more restricted in terms of their movement and therefore less motile, giving rise to lower diffusion at a population scale. For example, Cai et al. [9,10] use the exact form for $D(u)$ given in (4) in their studies of cell migration, explaining that this term models contact inhibition of cell locomotion. As part of their work, they fit their model to experimental data using fibroblast cells to estimate parameters, including the value of β in $D(u)$ [10].

The content of the letter is as follows. We summarise in Section 2.1 the nonclassical symmetry analysis that leads to (3). In Section 2.2 we explain how the solutions describe population extinction. Limits of large and small β are treated briefly in Section 2.3, noting the connection with Eq. (2), while a further illustrative example is provided in Section 2.4. We close in Section 3 with a discussion.

2. Derivation and interpretation

2.1. Nonclassical symmetry solution of the nonlinear reaction–diffusion equation

Classical Lie point symmetry analysis, first introduced by Sophus Lie, provides a systematic way to search for the invariant quantities in a differential equation, by seeking transformations that leave the equation of interest invariant. These classical symmetries can lead to the well-known travelling wave solutions or scale-invariant solutions (for more detail see for example [28,29]). Galaktionov, et al. [30] provided the first complete classical symmetry classification for equations of type (1).

The nonclassical symmetry (or Q-conditional) method was first introduced by Bluman and Cole [31], where we again seek transformations that leave the equation of interest invariant, but also require that the invariant surface condition be satisfied. This approach can sometimes result in additional symmetries that cannot be found using the classical method. If nonclassical symmetries can be found, they can be used in the same way as those found using the classical method. That is, the differential equation can be simplified and an analytic solution may be constructed. The most complete nonclassical symmetry analysis of equations of type (1) is given in Refs. [23,24].

To construct the exact solution (3) we use a nonclassical symmetry admitted by Eq. (1) whenever $R(u)$ and $D(u)$ are related by [23,24]

$$R(u) = \left(\alpha - \frac{\beta}{D(u)}\right) \int_{u^*}^{u} D(u')\,\mathrm{d}u', \tag{5}$$

where u^* is one of the zeros of $R(u)$ (here we choose $u^* = 0$) and α and β are constants. Eq. (1) can be reduced to the Helmholtz equation $\Psi_{xx} + \alpha\Psi = 0$, where $\Psi(x)$ is related to $u(x,t)$ by

$$\int_{u^*}^{u} D(u')\,\mathrm{d}u' = \mathrm{e}^{-\beta t}\,\Psi. \tag{6}$$

When $\alpha = k^2 > 0$, solutions to the Helmholtz equation can be written in terms of sine and cosine functions,

$$\Psi(x) = c_1 \sin kx + c_2 \cos kx, \tag{7}$$

where c_1 and c_2 are constants.

A pair of nonlinear reaction and diffusion terms that satisfy relationship (5) and are analogous to the Fisher–KPP equation are

$$D(u) = \frac{\beta(1-b)}{\alpha(u-b)}, \qquad R(u) = \beta(1-u)\ln\left(1 - \frac{u}{b}\right). \tag{8}$$

The solution of (1) with $D(u)$ and $R(u)$ given by (8) can be found by rearranging transformation (6) to obtain

$$u(x,t) = b\left[1 - \exp\left(\frac{\alpha}{\beta(b-1)}\exp(-\beta t)\,\Psi(x)\right)\right]. \tag{9}$$

2.2. One-parameter family of solutions

To reduce the number of parameters in (7) and (9), we are motivated by the linear problem [32]

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u, \quad 0 < x < L, \tag{10}$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad x = 0, \tag{11}$$

$$u = 0, \quad \text{on} \quad x = L, \tag{12}$$

$$u(x,0) = f(x), \quad 0 < x < L, \tag{13}$$

for which an exact solution can be found using separation of variables. For large time, this solution behaves like $u \sim A_1 \mathrm{e}^{-(\pi^2/4L^2-1)t}\cos(\pi x/2L)$, where A_1 is related to the initial condition (13). In the derivation of (10)–(13) from a dimensional system, the length-scale and time-scale of the original physical problem are related to the dimensional diffusion coefficient and growth rate parameter in the usual way so that the pde (10) does not contain any parameters. Apart from the initial condition (13), the only parameter in (10)–(13) is L. The final key point to note about (10)–(13) is that, because the sign of the eigenvalue $-\pi^2/4L^2+1$ dictates long-time behaviour, clearly the solution of (10)–(13) grows without bound if $L > \pi/2$ and decays (goes extinct) if $L < \pi/2$.

Returning to (1), the properties of solution (9) become much clearer if we choose the diffusion and reaction functions in (8) so that $D(u) \to 1$ and $R(u) \sim u$ as $u \to 0$. In other words, for small population density u, we want our nonlinear system to behave close to the linear system (10)–(13). As a result of this argument, we choose $b = -\beta$, $\alpha = 1+\beta$. Further, to match the boundary conditions (11)–(12), we choose $c_1 = 0$. Finally, to ensure $u(0,0) = u_0$, say, we choose $c_2 = -\beta \ln(1+u_0/\beta)$.

Before moving on, it is worth noting we are free to replace t with $t - t_0$ in (3), since the original pde (1) is invariant under translations in time. However, in that case the term $\ln(1+u_0/\beta)\mathrm{e}^{\beta t_0}$ is a constant, so changing t_0 is equivalent to redefining u_0. Therefore it is only worth keeping one of u_0 or t_0. We choose to keep the former.

With a single parameter β, we can illustrate the exact solutions with representative values of β. For example, we show in Fig. 2(a)–(c) the solutions (3) for $\beta = 0.1$, 1 and 10 (solid black curves), where the arrow indicates increasing time. There are qualitatively similar features in each case. For example, the solutions have the property $\partial u/\partial x = 0$ at $x = 0$, corresponding to no flux at the left boundary. Further, the solutions each have $u = 0$ at $x = \pi/(2\sqrt{1+\beta})$, which is a Dirichlet condition at the right boundary. Thus, physically speaking, there is a loss of mass at the right boundary and indeed u continues to decay until the population goes extinct as $t \to \infty$. Note the domain for (3) is decreasing in size as β increases. Also included in Fig. 2(a)–(c) are numerical solutions (green dashed), computed using finite differences with a no-flux and Dirichlet conditions at the left and right boundaries, respectively, with the details included on GitHub. Clearly there is a very good match, confirming the derivation of the exact solution. We return to Fig. 2 shortly.

2.3. Asymptotic limits

The regime $\beta \gg 1$ is interesting since, from (4),

$$D(u) \sim 1 - \frac{u}{\beta}, \quad R(u) \sim u(1-u) + \frac{u^2(1-u)}{2\beta}, \quad \text{as} \quad \beta \to \infty. \tag{14}$$

That is, for large β, to leading order the diffusion term is constant and the reaction term is logistic. Therefore, the reaction–diffusion Eq. (1) with (4) is a close approximation of the Fisher–KPP Eq. (2). In this limit,

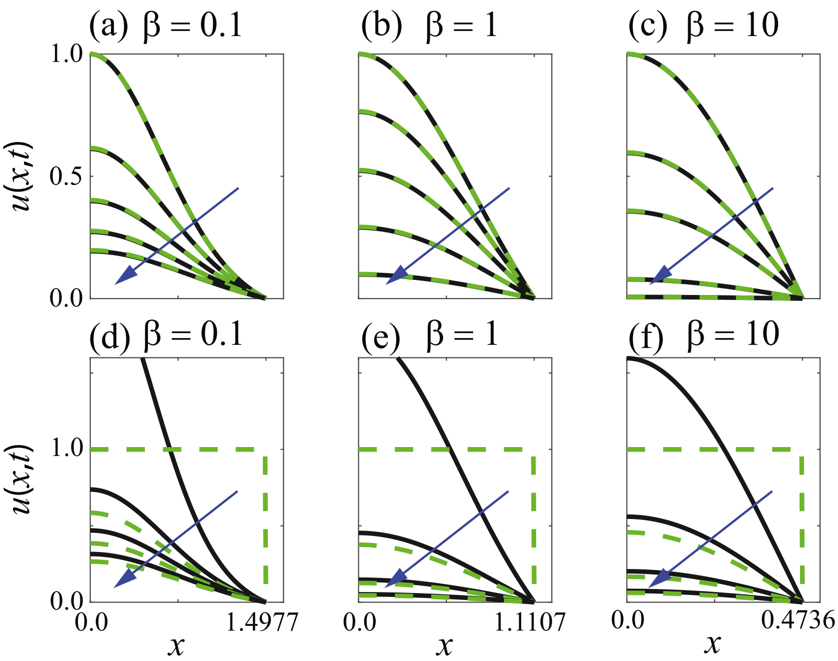

Fig. 2. Exact time-dependent solutions of (1). (a)–(c) Exact (solid black) and numerical (dashed green) solutions for $\beta = 0.1, 1$ and 10, respectively, with the arrows showing the direction of increasing t. Profiles in (a)–(c) are given at $\beta t = 0, 2, 4, 6$ and 8, $\beta t = 0, 0.2, 0.5, 1$ and 2, and $\beta t = 0, 1, 2, 5$ and 10, respectively. Profiles in (d)–(f) compare numerical solutions for $u(x, 0) = 1$ with the exact solutions (3) with u_0 chosen so that the initial mass is the same in each case. In (d)–(f) profiles are compared at $\beta t = 0, 4, 6$ and 8, $\beta t = 0, 1, 2$ and 3, and $\beta t = 0, 0.1, 0.2$ and 0.3. Numerical solutions are obtained using the method of lines where the x variable is discretised on a uniform mesh with 201 grid points.

from (3) we have $u \sim u_0 \mathrm{e}^{-\beta t} \cos\left(\sqrt{1+\beta}x\right)$ on $0 < x < \pi/(2\sqrt{1+\beta})$ as $\beta \to \infty$. Note the domain length continues to shrink as β increases.

In the limit $\beta \to 0^+$, we have $\pi/(2\sqrt{1+\beta}) \to \pi/2^-$, so the domain for the exact solution (3) when $\beta \ll 1$ is approximately $0 < x < \pi/2$. For $\beta \ll 1$, the nonlinear diffusivity $D(u)$ is not close to being constant, as we can see from Fig. 1(a). While the solution $u(x,t)$ decays very slowly for $\beta \ll 1$, ultimately for $t \gg 1/\beta$ the population density becomes small and $u \sim \beta \ln(1 + u_0/\beta)\, \mathrm{e}^{-\beta t} \cos\left(\sqrt{1+\beta}x\right)$ in this regime.

2.4. Illustrative example

As is common with exact solutions that come from symmetry analysis, we are not free to choose the initial condition in (3); instead, the solution at $t = 0$ must be

$$u(x,0) = \beta\left[\exp\left(\ln\left(1+\frac{u_0}{\beta}\right)\cos\left(\sqrt{1+\beta}\,x\right)\right) - 1\right]. \tag{15}$$

However, we are free to choose $u_0 = u(0, 0)$ to illustrate a simple but practical example. Suppose we are studying (1) and employ the obvious initial condition from a mathematical modelling perspective, namely $u(x, 0) = 1$ for $0 < x < L$. In this case the initial mass of the solution is L. With this in mind, we choose the value of u_0 in our exact solution (3) so the initial mass of the solution is $\pi/(2\sqrt{1+\beta})$ (remembering that $L = \pi/(2\sqrt{1+\beta})$ for (3)). By integrating (15) from $x = 0$ to $\pi/(2\sqrt{1+\beta})$, we arrive at the nonlinear algebraic equation

$$\frac{1}{\beta} + 1 = J_0\left(\ln\left(\frac{\beta+u_0}{\beta}\right)\right) + L_0\left(\ln\left(\frac{\beta+u_0}{\beta}\right)\right), \tag{16}$$

where $J_0(x)$ is the Bessel function of the first kind and $L_0(x)$ is a modified Struve function of the first kind. For a given β, we can solve (16) for u_0 numerically, for example using Newton's method. The result of these calculations is illustrated in Fig. 2(d)–(f). Here, numerical solutions (green dashed) computed for the physically interesting initial condition $u(x, 0) = 1$ are compared to the exact solution with u_0 found from (16). While there is no match for small time, we argue that the exact solutions (3) provide reasonably good approximations for the numerical solutions for intermediate to large times.

3. Discussion

In this article we have analysed a one-parameter family of exact solutions (3) of reaction–diffusion model (1), where the nonlinear diffusion and reaction terms are given by (4). To put this work into context, these exact solutions are qualitatively similar to solutions of the well-known Fisher–KPP model

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + u(1-u), \quad 0 < x < L, \tag{17}$$

$$\frac{\partial u}{\partial x} = 0 \quad \text{on} \quad x = 0, \tag{18}$$

$$u = 0, \quad \text{on} \quad x = L, \tag{19}$$

$$u(x, 0) = f(x), \quad 0 < x < L, \tag{20}$$

provided $L < \pi/2$. This correspondence is possible because both $R(u)$ and the logistic growth term $u(1-u)$ are monostable with roots at $u = 0$ and $u = 1$ and both functions are asymptotic to u as $u \to 0$. Indeed, for the one-parameter family of exact solutions (3)–(4) and all solutions to the Fisher–KPP model (17)–(20), we have

$$u \sim U_1 \mathrm{e}^{-(\pi^2/4L^2 - 1)t} \cos\left(\frac{\pi x}{2L}\right) \quad \text{as} \quad t \to \infty, \tag{21}$$

where L is: any constant $0 < L < \pi/2$ for the Fisher–KPP model; related to $\beta > 0$ for our exact solutions (3) via $L = \pi/(2\sqrt{1+\beta})$. Further, the constant U_1 in (21) is: related to $f(x)$ in (20) in Fisher–KPP in a complicated way; or given by $U_1 = \beta \ln(1 + u_0/\beta)$ for our exact solutions (3). Therefore, in this sense, the exact solutions (3) are perfectly sensible and consistent with previous understanding of these types of reaction–diffusion models (including the extinction property $L < \pi/2$, for example [33–35]). Finally, an additional connection is that (1) with (4) approaches the Fisher–KPP equation in the limit $\beta \to \infty$, as indicated by (14).

There is the usual downside that comes from symmetry analysis, which is that we are not free to change the initial condition (15) to suit the physical application or experimental data, for example. Regardless, (15) is a decreasing function with properties $u(0, 0) = u_0$ and $u(L, 0) = 0$, which is certainly reasonable, provided $u_0 = \mathcal{O}(1)$. Of course, what makes the family of exact solutions (3) exceptional and worth recording is not whether they match a particular initial condition, but rather that fully explicit time-dependent solutions to nonlinear reaction–diffusion models are very rare. Further, apart from explicitly showing how the solution evolves, these formulae can also be used as benchmarks for numerical simulations, for example by researchers or in undergraduate courses for numerical methods in nonlinear pdes.

We close by mentioning that the particular nonclassical symmetry used to construct solution (9) is valid in any number of dimensions and in any coordinate system [36], so that solutions to Eq. (1) with nonlinear diffusivity and reaction given by (8) could be constructed in $\mathbb{R}^n$. For example, in radially symmetric coordinates in $\mathbb{R}^2$, the solution (9) will still be valid, however the solution of the Helmholtz equation, $\Psi(r)$, will be written in terms of Bessel functions and hold on $L < \lambda/\sqrt{1+\beta}$, where λ is the first zero of the Bessel function $J_0(x)$ (in $\mathbb{R}^3$ the solution will be in terms of spherical Bessel functions [27]). The higher dimensional analogue of many of the other results presented here can also be produced, for example the solution will decay and become extinct as $t \to \infty$.

Acknowledgements

This work is supported by the Australian Research Council (DP200100177, DP200102130). The authors thank the Australian Mathematical Sciences Institute for funding the Workshop, "PDEs in Mathematical Biology" hosted jointly at QUT and UniSA in June 2021, which motivated the current research.

Appendix A. Supplementary data

Supplementary material is available at GitHub.

References

[1] R.A. Fisher, The wave of advance of advantageous genes, Ann. Eugen. 7 (1937) 355–369.
[2] A.N. Kolmogorov, P.G. Petrovskii, N.S. Piskunov, A study of the diffusion equation with increase in the amount of substance, and its application to a biological problem, Moscow Univ. Math. Bull. 1 (1937) 1–26.
[3] V. Melica, S. Invernizzi, G. Caristi, Logistic density-dependent growth of an *Aurelia aurita* polyps population, Ecol. Model. 291 (2014) 1–5.
[4] M.J. Simpson, A.P. Browning, D.J. Warne, O.J. Maclaren, R.E. Baker, Parameter identifiability and model selection for sigmoid population growth models, J. Theor. Biol. 535 (2022) 110998.
[5] S.T. Vittadello, S.W. McCue, G. Gunasingh, N.K. Haass, M.J. Simpson, Mathematical models for cell migration with real–time cell cycle dynamics, Biophys. J. 114 (2018) 1241–1253.
[6] J.A. Sherratt, J.D. Murray, Models of epidermal wound healing, Proc. R. Soc. Lond. Ser. B 241 (1990) 29–36.
[7] W. Jin, C.J. Penington, S.W. McCue, M.J. Simpson, Stochastic simulation tools and continuum models for describing two-dimensional collective cell spreading with universal growth functions, Phys. Biol. 13 (2016) 056003.
[8] S.W. McCue, W. Jin, T.J. Moroney, K.-Y. Lo, S.-E. Chou, M.J. Simpson, Hole-closing model reveals exponents for nonlinear degenerate diffusivity functions in cell biology, Phys. D Nonlinear Phenom. 398 (2019) 130–140.
[9] A.Q. Cai, K.A. Landman, B.D. Hughes, Modelling directional guidance and motility regulation in cell migration, Bull. Math. Biol. 68 (2006) 25–52.
[10] A.Q. Cai, K.A. Landman, B.D. Hughes, Multi-scale modeling of a wound-healing cell migration assay, J. Theor. Biol. 245 (2007) 576–594.
[11] M. Ablowitz, A. Zeppetella, Explicit solutions of Fisher's equation for a special wave speed, Bull. Math. Biol. 41 (1979) 835–840.
[12] P. Kaliappan, An exact solution for travelling waves of $u_t = Du_{xx} + u - u^k$, Phys. D Nonlinear Phenom. 11 (1983) 368–374.
[13] S.W. McCue, M. El-Hachem, M.J. Simpson, Exact sharp–fronted travelling wave solutions of the Fisher–KPP equation, Appl. Math. Lett. 114 (2021) 106918.
[14] Y. Kametaka, On the nonlinear diffusion equation of Kolmogorov-Petrovskii-Piskunov type, Osaka J. Math. 13 (1976) 11–66.
[15] H.P. McKean, Nagumo's equation, Adv. Math. 4 (1970) 209–223.
[16] J. Rinzel, Neutrally stable travelling wave solutions of nerve conduction equations, J. Math. Biol. 2 (1975) 205–217.
[17] G.A. Carpenter, Periodic solutions of nerve impulse equations, J. Math. Anal. Appl. 58 (1977) 152–173.
[18] S. Hastings, The existence of periodic solutions to Nagumo's equations, Q. J. Math. 25 (1974) 369–378.
[19] D.J. Arrigo, J.M. Hill, P. Broadbridge, Nonclassical symmetry reductions of the linear diffusion equation with a nonlinear source, IMA J. Appl. Math. 52 (1994) 1–24.
[20] P.A. Clarkson, E.L. Mansfield, Symmetry reductions and exact solutions of a class of nonlinear heat equations, Physica D 70 (1994) 250–288.
[21] R. Conte, Universal invariance properties of painlevé analysis and bäcklund transformation in nonlinear partial differential equations, Phys. Lett. A 134 (1988) 100–104.
[22] Z.X. Chen, B.Y. Guo, Analytic solutions of the Nagumo equation, IMA J. Appl. Math. 48 (1992) 107–115.
[23] D.J. Arrigo, J.M. Hill, Nonclassical symmetries for nonlinear diffusion and absorption, Stud. Appl. Math. 94 (1995) 21–39.
[24] J.M. Goard, P. Broadbridge, Nonclassical symmetry analysis of nonlinear reaction–diffusion equations in two spatial dimensions, Nonlinear Anal. Theory Methods Appl. 26 (1996) 735–754.
[25] P. Broadbridge, B.H. Bradshaw-Hajek, Exact solutions for logistic reaction–diffusion equations in biology, Z. Angew. Math. Phys. 67 (2016) 1–13.
[26] B.H. Bradshaw-Hajek, P. Broadbridge, Analytic solutions for calcium ion ferilisation waves on the surface of eggs, Math. Med. Biol. J. IMA 36 (2019) 549–562.
[27] B.H. Bradshaw-Hajek, P. Broadbridge, An analytic solution for a Gompertz-like reaction–diffusion model for tumour growth, in: The Role of Metrics in the Theory of Partial Differential Equations, Vol. 1, 2020, pp. 127–136.
[28] P.J. Olver, Applications of Lie Group to Differential Equations, Vol. 107, AMS, Springer, New York, 1982.

[29] N.H. Ibragimov, CRC Handbook of Lie Group Analysis of Differential Equations, Volume I: Symmetries, Exact Solutions, and Conservation Laws, CRC Press, Boca Raton, 1994.

[30] V.A. Galaktionov, V.A. Dorodnitsyn, G.G. Elenin, S.P. Kurdyumov, A.A. Samarskii, A quasilinear heat equation with a source: peaking, localization, symmetry exact solutions, asymptotics, structures, J. Sov. Math. 41 (1988) 1222–1292.

[31] G.W. Bluman, J.D. Cole, General similarity solution of the heat equation, J. Math. Mech. 18 (1969) 1025–1042.

[32] J.G. Skellam, Random dispersal in theoretical populations, Biometrika 38 (1951) 196–218.

[33] E. Bradford, J.R. Philip, Stability of steady distributions of asocial populations dispersing in one dimension, J. Theor. Biol. 29 (1970) 13–26.

[34] M. El–Hachem, S.W. McCue, W. Jin, Y. Du, M.J. Simpson, Revisiting the Fisher-Kolmogorov-Petrovsky-Piskunov equation to interpret the spreading–extinction dichotomy, Proc. R. Soc. A 475 (2019) 20190378.

[35] Y. Li, S.T. Johnston, P.R. Buenzli, P. van Heijster, M.J. Simpson, Extinction of bistable populations is affected by the shape of their initial spatial distribution, Bull. Math. Biol. 84 (2022) 21.

[36] P. Broadbridge, B.H. Bradshaw-Hajek, D. Triadis, Exact non-classical symmetry solutions of Arrhenius reaction–diffusion, Proc. R. Soc. A 471 (2015) 20150580.